

Example #1: Proof by Contraposition

Conjecture: If $ac \leq bc$, then $c \leq 0$, when $a > b$.

Proof (Contrapositive): Assume $c > 0$, and $a > b$,
and show $bc < ac$.

Multiplying both sides of $a > b$ by c
gives $ac > bc$.

Therefore, if $ac \leq bc$, then $c \leq 0$, when $a > b$.

Example #2: Proof by Contraposition

Conjecture: If n^2 is even, then n is even.

Proof (Contraposition): Assume n is odd, show
 n^2 is odd.

$$n = 2k + 1, k \in \mathbb{Z}.$$

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

This shows that n^2 has the form $2m + 1$, $m \in \mathbb{Z}$,
and so n^2 is odd.

$\therefore$ if n^2 is even, then n is even.

Proof by Contradiction

(a.k.a. *Reductio ad Absurdum*)

Recall the Law of Implication: $p \rightarrow q \equiv \neg p \vee q$

$$\overline{p \rightarrow q} \equiv \overline{\neg p \vee q} \equiv \overline{\neg p} \wedge \overline{q} \equiv p \wedge \overline{q}$$

$$\text{want : } p \wedge \overline{q} \equiv F$$

$$\text{Assume : } p \wedge \overline{q}$$

Show: a contradiction

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Example #1: Proof by Contradiction

Conjecture: If $3n + 2$ is odd, then n is odd.

Proof (Contradiction): Assume $3n + 2$ is odd,
and n is even.

$$\begin{aligned} \text{So } n = 2k, k \in \mathbb{Z}. \quad 3n + 2 &= 3(2k) + 2 = 6k + 2 \\ &= 2(3k + 1). \end{aligned}$$

We have shown that $3n + 2$ is even, which contradicts the assumption that $3n + 2$ is odd.

$\therefore$ if $3n + 2$ is odd, then n is odd.

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Example #2: Proof by Contradiction (1 / 2)

Conjecture: The sum of the squares of two odd integers is never a perfect square. (Or: If $n = a^2 + b^2$, then n is not a perfect square, where $a, b \in \mathbb{Z}^{\text{odd}}$.)

Proof (Contradiction) : Assume $n = a^2 + b^2$, and n is a perfect square, where $a, b \in \mathbb{Z}^{\text{odd}}$

$$a = 2k+1, k \in \mathbb{Z}. \quad b = 2j+1, j \in \mathbb{Z}.$$

$$n = (2k+1)^2 + (2j+1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 4k^2 + 4k + 4j^2 + 4j + 2 = 2(2k^2 + 2k + 2j^2 + 2j + 1).$$

So, n is even, because $n = 2m$, $m \in \mathbb{Z}$.

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Example #2: Proof by Contradiction (2 / 2)

n is a perfect square. $n = q^2$, $q \in \mathbb{Z}$.

Since I have shown that n is even, so q^2 is even.

Because of our previous proof, we know that q is even.

Since q is even, $q = 2i$, $i \in \mathbb{Z}$. So $q^2 = 4i^2$,

so $n = 4i^2$. So n is a multiple of 4.

But previously, n was only a multiple of 2.

That is a contradiction.

$\therefore$ if $n = a^2 + b^2$, then n is not a perfect square, where $a, b \in \mathbb{Z}^{\text{odd}}$

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How To Prove Biconditional Expressions

(i.e., Conjectures Of The Form $p \leftrightarrow q$)

$$p \leftrightarrow q \equiv p \rightarrow q \wedge q \rightarrow p$$

So to prove $p \leftrightarrow q$, we want to prove $p \rightarrow q$ and $q \rightarrow p$, using any of the methods we've discussed.

Example(s):

We've already done one!

$$n \in \mathbb{Z}^{\text{even}} \rightarrow n^2 \in \mathbb{Z}^{\text{even}}$$

$$n^2 \in \mathbb{Z}^{\text{even}} \rightarrow n \in \mathbb{Z}^{\text{even}}$$

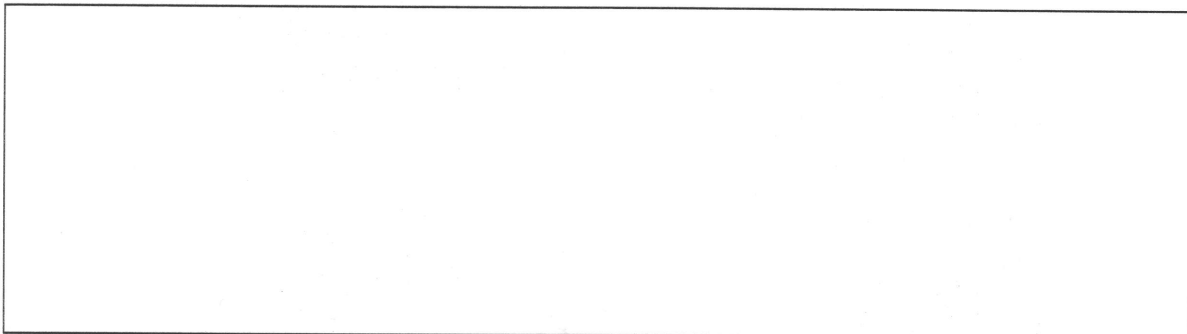
$$\text{so } n \in \mathbb{Z}^{\text{even}} \leftrightarrow n^2 \in \mathbb{Z}^{\text{even}}$$

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Disproving Conjectures

Two common approaches:

Example(s):



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