Example #1: Proof by Contraposition

Conjecture: If $ac \le bc$, then $c \le 0$, when $a > b$.

Proof (contrapositive): Assume $c > 0$, and $a > b$, and show $bc < ac$.

Multiplying both sides of $a > b$ by c gives $ac > bc$.

Therefore, if $ac \le bc$, then $c \le 0$, when $a > b$.

Example #2: Proof by Contraposition

Conjecture: If n^2 is even, then n is even.

Proof (Contraposition): Assume n is odd, show n^2 is odd.

$n = 2k+1$, $k \in \mathbb{Z}$.

$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

This shows that n^2 has the form $2m+1$, $m \in \mathbb{Z}$, and so n^2 is odd.

$\therefore$ if n^2 is even, then n is even.

Proof by Contradiction

(a.k.a. Reductio ad Absurdum)

Recall the Law of Implication: $p \to q \equiv \neg p \vee q$

$\overline{p \to q} \equiv \overline{\bar{p} \vee q} \equiv \bar{\bar{p}} \wedge \bar{q} \equiv p \wedge \bar{q}$

want : $p \wedge \bar{q} \equiv F$

Assume : $p \wedge \bar{q}$

Show : a contradiction

Example #1: Proof by Contradiction

Conjecture: If $3n+2$ is odd, then n is odd.

Proof (contradiction): Assume $3n+2$ is odd, and n is even.

So $n = 2k$, $k \in \mathbb{Z}$. $3n+2 = 3(2k)+2 = 6k+2$
$= 2(3k+1)$.

We have shown that $3n+2$ is even, which contradicts the assumption that $3n+2$ is odd.

$\therefore$ if $3n+2$ is odd, then n is odd.

Example #2: Proof by Contradiction (1 / 2)

Conjecture: The sum of the squares of two odd integers is never a perfect square. (Or: If $n = a^2 + b^2$, then n is not a perfect square, where $a, b \in \mathbb{Z}^{odd}$.)

Proof (Contradiction): Assume $n = a^2 + b^2$, and n is a perfect square, where $a, b \in \mathbb{Z}^{odd}$

$a = 2k+1, k \in \mathbb{Z}$. $b = 2j+1, j \in \mathbb{Z}$.

$n = (2k+1)^2 + (2j+1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 =$
$4k^2 + 4k + 4j^2 + 4j + 2 = 2(2k^2 + 2k + 2j^2 + 2j + 1)$.

So, n is even, because $n = 2m$, $m \in \mathbb{Z}$.

Example #2: Proof by Contradiction (2 / 2)

n is a perfect square. $n = q^2$, $q \in \mathbb{Z}$.

Since I have shown that n is even, so q^2 is even.

Because of our previous proof, we know that q is even.

Since q is even, $q = 2i$, $i \in \mathbb{Z}$. So $q^2 = 4i^2$,

So $n = 4i^2$. So n is a multiple of 4.

But previously, n was only a multiple of 2.

That is a contradiction.

$\therefore$ if $n = a^2 + b^2$, then n is not a perfect square, where $a, b \in \mathbb{Z}^{odd}$

How To Prove Biconditional Expressions

(i.e., Conjectures Of The Form $p \leftrightarrow q$)

$p \leftrightarrow q \equiv p \to q \land q \to p$

So to prove $p \leftrightarrow q$, we want to prove $p \to q$ and $q \to p$, using any of the methods we've discussed.

Example(s):

We've already done one!

$n \in \mathbb{Z}^{even} \to n^2 \in \mathbb{Z}^{even}$

$n^2 \in \mathbb{Z}^{even} \to n \in \mathbb{Z}^{even}$

So $n \in \mathbb{Z}^{even} \leftrightarrow n^2 \in \mathbb{Z}^{even}$

Disproving Conjectures

Two common approaches:

Example(s):