C SC 245 - Introduction to Discrete Structures (Fall 2010)

Section #11

November 2-3, 2010

1 Sequences and Summations

1. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals

   (a) $a_0 = 2, a_1 = 3, a_2 = 5, a_3 = 9$

   (b) $a_0 = 1, a_1 = 4, a_2 = 9, a_3 = 16$

   (c) $a_0 = 0, a_1 = 0, a_2 = 1, a_3 = 1$

   (d) $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3$

2. What are the values of these sums?

   (a) $2 + 3 + 4 + 5 + 6 = 20$

   (b) $1 + (-2) + 4 + (-8) + 16 = 11$

   (c) $3 * 10 = 30$

   (d) $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 = 511$

3. What is the value of each of these sums of terms of a geometric progression?

   (a) $3(1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256) = 1533$

   (b) $2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 = 510$

   (c) $(-3)^2((-3)^0 + (-3)^1 + (-3)^2 + (-3)^3 + (-3)^4 + (-3)^5 + (-3)^6) = 4923$

   (d) $2(\frac{4923}{9} + (-3)^7 + (-3)^8) = 9842$

   Note: You can also use the formula for the sum of a geometric progression.
   (e.g. $\sum_{j=0}^{n} ar^j = a(r^{n+1} - 1)/(r - 1)$)

4. Show that $\displaystyle\sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - a_0$, where $a_0, a_1, \ldots, a_n$ is a sequence of real numbers. This type of sum is called **telescoping.**

   For this, all we need to do is write out the meaning of the summation, and we will see that terms in the sequence cancel.

$$\sum_{j=1}^{n}(a_j - a_{j-1}) = a_1 - a_0 + a_2 - a_1 + \cdots + a_{n-1} - a_{n-2} + a_n - a_{n-1} = a_n - a_0$$

5. Find $\sum_{k=100}^{200} k$.

Note that the closed-form expression of $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Here, we have

$$\begin{aligned}\sum_{k=100}^{200} k &= \sum_{k=1}^{200} k - \sum_{k=1}^{99} k \\ &= \frac{200^2+20}{2} - \frac{99^2+99}{2} \\ &= 20100 - 4950 = 15150\end{aligned}$$

6. Find $\sum_{j=0}^{4} j!$.

By definition of factorial, we have

$$\begin{aligned}0!+1!+2!+3+4! &= 1+1+1\cdot 2+1\cdot 2\cdot 3+1\cdot 2\cdot 3\cdot 4 \\ &= 1+1+2+6+24 = 34\end{aligned}$$

7. Determine whether each of these sets is countable or uncountable. For those that are countable, exhibit a one-to-one correspondence between the set of natural numbers and that set.

(a) Countable. The Correspondence matches the natural number n with the string consisting of n ones

(b) Countable. This set is a subset of the rational numbers. Apply the demonstration from Example 20.

(c) Uncountable. Apply the diagonal argument of Example 21.

(d) Uncountable. Apply the diagonal argument of Example 21.

2 Mathematical Induction

8. Prove that if $h > -1$, then $1 + nh \leq (1 + h)^n$ for all nonnegative integers n. This is called **Bernoulli's inequality**.

Proof. (By weak induction on $k \in \mathbb{Z}^*$)

Basis Step:
Let $k = 0$. For our conjecture, we have

$$1 + (0)h = 1 = (1 + h)^0$$

Inductive Step:
Assume $1 + kh \leq (1 + h)^k$ for some nonnegative integer k. We will show that $1 + (k + 1)h \leq (1 + h)^(k + 1)$.

$$\begin{aligned}(1 + h)^{k+1} &= (1 + h)^k(1 + h)\\ &\geq (1 + kh)(1 + h) \text{ (By the inductive hypothesis)}\\ &= 1 + h + kh + kh^2\\ &= 1 + (k + 1)h + kh^2\end{aligned}$$

By comparing like terms as well as noting that h^2 is always positive and $h + 1 > 0$, we have

$$1 + (k + 1)h \leq 1 + (k + 1)h + kh^2 \leq (1 + h)^{k+1} \text{ (By transitivity of } \leq)$$
$$1 + (k + 1)h \leq (1 + h)^{k+1}$$

Therefore, if $h > -1$, then $1 + nh \leq (1 + h)^n$ for all nonnegative integers n. □

9. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B).$$

Proof. (By weak induction on $k \in \mathbb{Z}^+$)

Basis Step:
Let $k = 1$. For our conjecture, we have $(A_1) \cap B = A_1 \cap B$.

Inductive Step:
Assume $(A_1 \cup \cdots \cup A_k) \cap B = (A_1 \cap B) \cup \cdots \cup (A_k \cap B)$ for some positive integer k.
We will show that $(A_1 \cup \cdots \cup A_k \cup A_{k+1}) \cap B = (A_1 \cap B) \cup \cdots \cup (A_k \cap B) \cup (A_{k+1} \cap B)$.

$$\begin{aligned}(A_1 \cap B) \cup \cdots \cup (A_k \cap B) \cup (A_{k+1} \cap B) &= ((A_1 \cup \cdots \cup A_k) \cap B) \cup (A_{k+1} \cap B) \text{ (By the inductive hypothesis)} \\ &= (A_1 \cup \cdots \cup A_k \cup A_{k+1}) \cap B \text{ (By the distributive property)}\end{aligned}$$

Therefore, if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B).$$

□