Cornell University Department of Physics

Physics 2213: Electromagnetism, Fall 2012

Problem Set # 2

(Due Friday, September 7 at 5:00pm *sharp*.)

Agenda and readings

Readings marked YF refer to sections from the text book, *University Physics*, 13th edition, volume 2, by Young and Freedman.

- **NOTE: Lab #1 starts THIS week!!!**
- Lec 4 9/4: Fluid model, electric flux, Gauss's law **(YF 22.1-3)**
- Lec 5 9/6: Gauss's Law & Applications **(YF 22.4)**

To help identify what this assignment asks of you, the questions you are asked are marked with question marks '?', additional things you should do are marked with underlines, *hints* appear in *italics*, and **important information** appears in **boldface**.

Contents

1 Electric field concept

As discussed in lecture, the *electric field* at any point $\vec{x}$ is defined as the force per unit charge that a charge would feel when placed at that point, *provided all other charges are held fixed in magnitude, sign, and location.* This definition is convenient mathematically because the computation of the field at point $\vec{x}$ does not depend on the magnitude of the charge that would be placed there. Also, frequently, in physical arguments, the magnitude of that imagined charge cancels out so that only the electric field is relevant. We saw an example of this in Problem 3 from Problem Set # 1. Let's briefly consider how this problem would look if we considered electric fields only. Figure 1 shows the arrangement of charges we considered on the last problem set, where to make our first estimate of what happens in this situation, we assume that all the negative charge $-q$ built up in the conductor collects that the point nearest the external charge $+Q$ and that all of the remaining positive charge $+q$ collects at the point furthest from the external charge $+Q$.

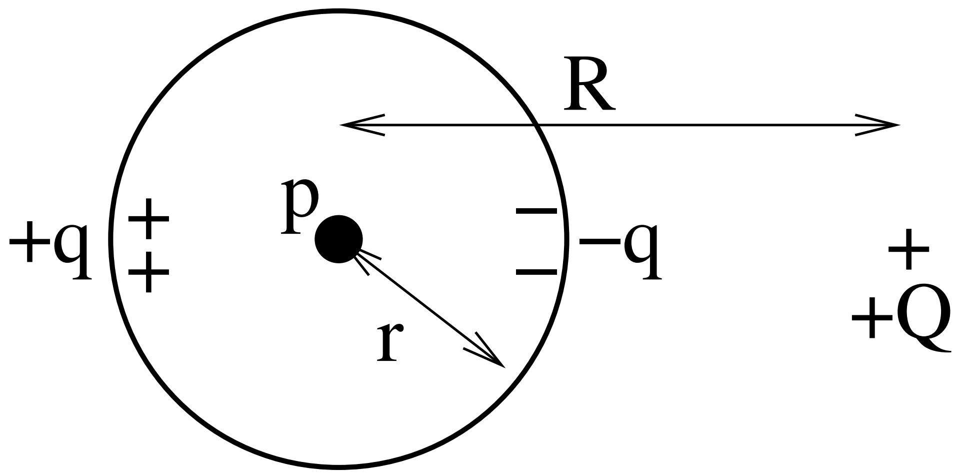

Figure 1: Model for polarization response of a solid spherical conductor near a positive external charge $+Q$
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1.1 Electric field at the center of the sphere

Instead of placing an actual charge p at the center of the sphere, consider the electric field at the center of the sphere. In the absence of any charge at the center of the sphere, what is the electric field at the sphere's center, in terms of no quantities other than R, r, q, Q, and k_c?

1.2 Condition for equilibrium

If we want to ensure that none of the mobile charges at the center of the sphere would want to move under equilibrium conditions where all the charges have finally stopped moving, (a) what would the value of the electric field $\vec{E}$ have to be at the center of the sphere?

Finally, (b) use your condition on $\vec{E}$ to solve for the value of the polarization charge q which builds up on the sphere at equilibrium.

Lessons: The point of these parts has been to show you that working with electric fields really amounts to the same thing as working with forces, but with the advantage of not having to worry about the magnitude of hypothetical charges you might use in a physical argument, like the factors representing the charge p from Problem 3 on the last problem set.

1.3 Electric field conditions in conducting materials

The conditions for electrostatic in conductors are most naturally expressed in terms of electric fields. Based on the arguments we have been using above, (a) what must the value of the electric field be anywhere inside a conducting material under equilibrium conditions? And, (b) explain (in a brief sentence or two) why this must be true *in general* for a conductor of any shape, carrying any net charge, and with any arrangement of charges around it.

1.4 Units of electric field

Because the electric field is the *force* on an imagined charge *per unit charge*, electric fields are measured in units of newton/coulomb≡N/C. There is a different, and much more common way, to express the units of electric fields.

To understand this equivalent unit, consider the situation in Figure 2, which similar to what we saw in the example of the electric vehicle from the previous problem set. As discussed in Problem Set #1, charge in this situation will flow from the bottom plate (leaving it with a net negative charge) to the top plate (giving it a positive charge) until the amount of energy needed to move 1 C of charge from the bottom plate to the top plate is 100 J. (Recall that 1 volt=1 joule/coulomb.)

Figure 2 below shows a situation from the electric car problem from Problem Set #1.

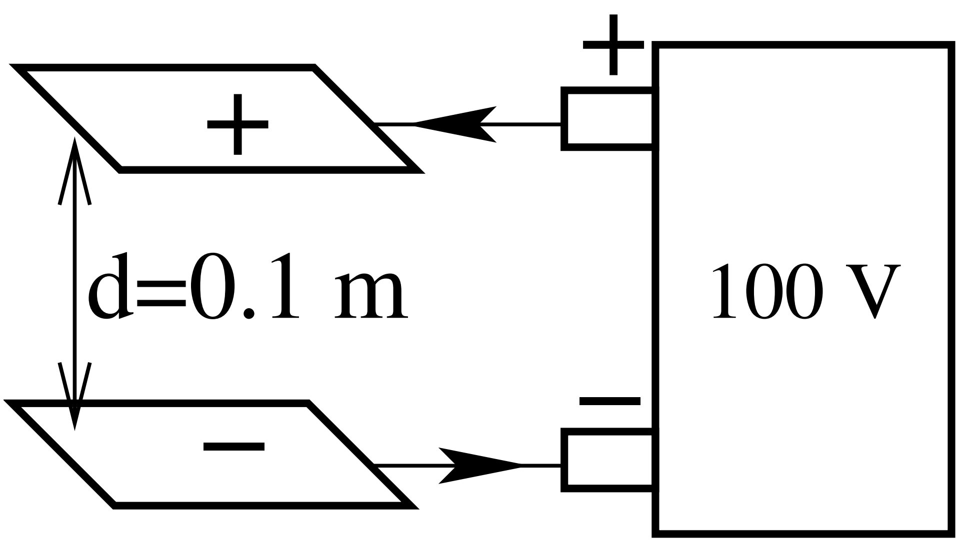

Figure 2: Charging of plates by a battery

Suppose that, now, a charge of +1 C is brought into the system just above the negative plate and then is forced toward the positive plate along the path indicated by the arrow until it is just below the positive plate. Clearly, the positive charge of +1 C will not like to be forced toward the top plate with its positive charge (nor moved away from the negative plate to which it is attracted), and there will be a force opposing this motion. Given that the work needed to move the +1 C along the arrow in this situation is 100 J, (a) what is the average force on the charge while it is moved? And, (b) what then is the average electric field between the plates?

Now, suppose that the voltage of the battery was V and the distance between the plates was d. Then, (c) what would the electric field between the plates be in terms of V and d?

Hint: Your answer should help you understand why electric fields are often measured in units of volts/meter.

1.5 Electric field in MOSFET field-effect transistor

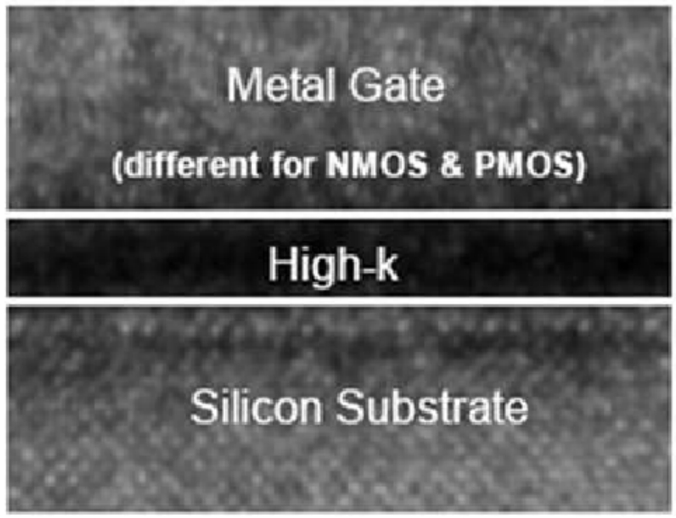

Figure 3: High resolution TEM of gate in microelectronic transistor. "High-k" region (corresponding to parallel plates) is so narrow that individual rows of atoms can be seen in the silicon.

The electric fields in the components of micro-electronics (as in your cell phone, tablet or computer) can actually be amazingly high because of the extremely small dimensions of the circuit elements. Figure 3 shows a very high resolution *transmission* electron micrograph (TEM) of a typical transistor. The region labeled "High-k" in the figure corresponds to an edge-on view of the two parallel plates in Figure 2. The little blobs in the region labeled "Silicon Substrate" actually correspond to atoms, which means that the separation between the plates is only a few layers, in this case about 1.5 nm (1.5×10^{-9} m)! Typical operating voltages for such transistors are about 0.5 V. From this information, estimate the electric field across the "High-k" dielectric.
Hint: Your answer should come out less than the breakdown field of the gate material, which is 0.5 GV/m, but not too much less, because these devices are generally engineered right at the edge of the very limitations of the materials used.

2 Electric fields in vector situations

Figure 4 shows an arrangement of two charges (of charge ∓ 0.5 nC, respectively) at a distance of 6 m from each other. The figure also shows the coordinate system you are to use. (Note that this arrangement of displaced charges of equal magnitude but opposite sign is precisely the same type of arrangement we looked at when considering the polarization of metal spheres. This type of arrangement is generally referred to as a "dipole".)

2.1 Fields at location of solid boxes

Compute the x- and y- components and magnitude of the resulting electric field at each of the points (a-d) marked with solid boxes in the figure.

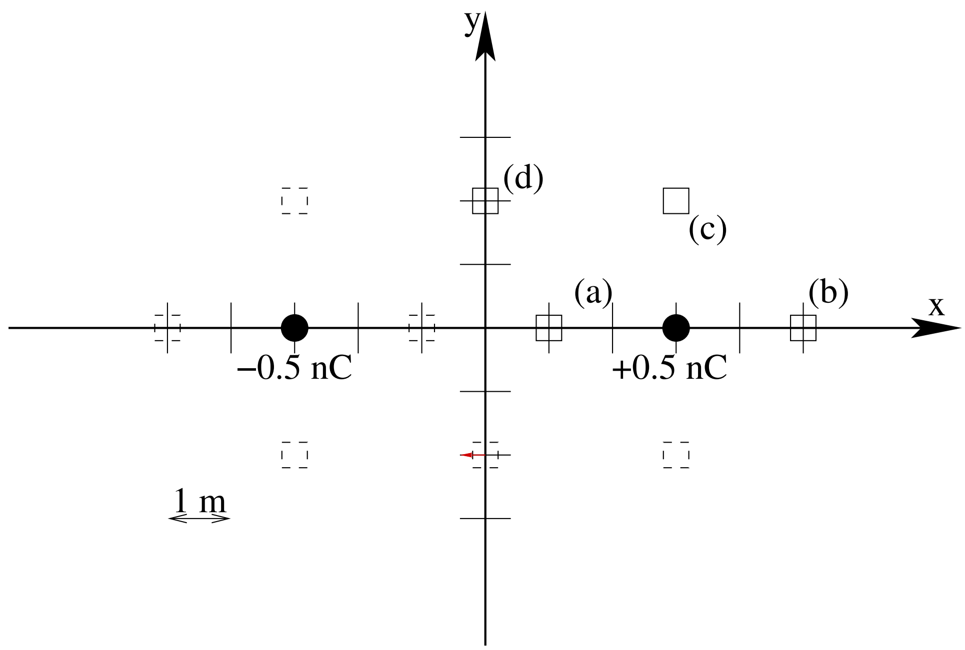

Figure 4: Space near and around dipole arrangement of charges.

2.2 Field map

For each of the points marked with boxes (either solid or dashed), (a) sketch the electric field as the corresponding vector $\vec{E}$ with the base of the vector at the corresponding point and with the tic marks on the axes representing 1 N/C $\equiv$ 1 V/m, following the example given for the point ($x = 0, y = -2$ m).

Then, (b) complete the map by sketching the field vectors at the locations of the dashed boxes.

Important *hint*: To finish the sketch, you should not have to do any new detailed calculations. For each type of point, all the contributions will be the same, just the various signs will be different. This means that the magnitudes of the electric field at points of the same type will be the same, and you will only need to figure out the direction of the total field at each point.

3 Electric dipoles

The particular charge pattern that appeared in our analysis of the polarization of the metal sphere (Figure 1, namely equal in magnitude but opposite charges at a certain separation, is a very important configuration known as a *dipole* ("di" because two types of charges, + and −, are involved, and "pole" for polarity.) Such configurations are quite important in nature because it takes a lot of energy (as we have learned) to put a significant net charge on any object, and objects with a net charge over time generally attract opposite charges to themselves and ultimately become neutral. The type of charge separation we saw in the metallic sphere, on the other hand, is generally much easier to attain in nature.

Another common, and quite important, example of dipolar charge arrangements occur in molecules. As we just mentioned, molecules tend to be neutral whenever possible (although ionized species certainly exist, particularly in solutions), but generally the atoms in them carry different charges. A classic example is the water molecule, where the oxygen atom carries a charge of nearly −2, with each proton carrying a charge near +1 so that the molecule is electrically neutral.

Electric dipoles have a number of simple but important properties and applications. (In fact, in Problem 2, you sketched the electric field in the vicinity of an electric dipole.) In this problem, you will explore those properties using what you already know about electric fields and charges.

In this problem we will consider a general dipole with charges $\pm q$ separated by a distance Δ. To help make the mathematics work out more easily, we choose our coordinate system as in Figure 5, with the origin of our coordinate system at the location of the $-q$ charge, and the x-axis oriented along the line from the $-q$ to the $+q$ charge.

At the end of the day, you will find that all of your answers can be written in terms of a single quantity that summarized the property of the dipole, a special vector $\vec{p} \equiv q\Delta\hat{\imath}$ called the "dipole moment".

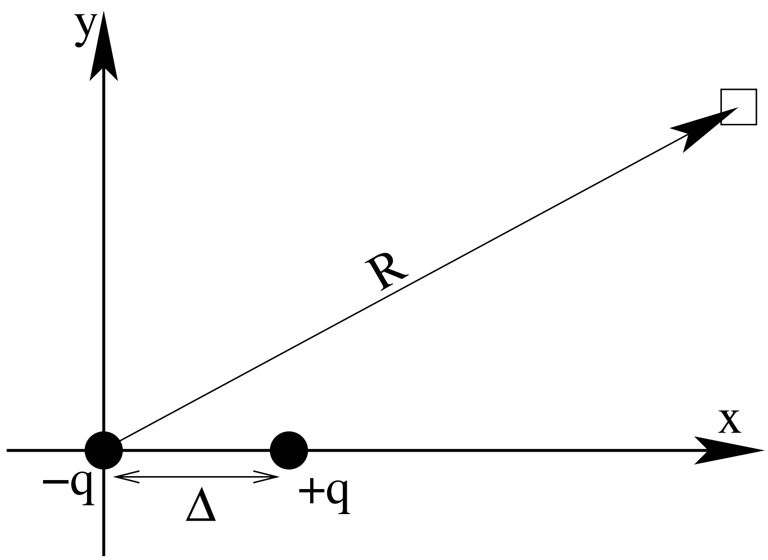

Figure 5: Electric dipole

3.1 Force and torque on an electric dipole in a *constant* electric field.

Suppose this dipole is placed in the presence of an electric field $\vec{E}$ which is constant everywhere in space with value $\vec{E}_0$. (I.e., $\vec{E}(x,y,z) = \vec{E}_0$).

As a specific example, let $q = 1$ nC and $\Delta = 0.01$ m (1 cm), and suppose the electric field is along the $+y$–axis with magnitude $E_0 = 10{,}000$ V/m. What is (a) the total force $\vec{F}_{\text{tot}} \equiv \sum_i \vec{F}_i$ on the dipole? And, what is the total torque $\vec{\tau} \equiv \sum_i \vec{r}_i \times \vec{F}_i$ on the dipole, when computed about (b) the origin, (c) the center (midpoint) of the dipole, and (d) the location of the positive charge? (e) In what direction does the torque try to align the dipole – with the dipole aligned with the field (i.e., the line from the $-q$ to the $+q$ charges along the direction of the field), with the dipole aligned perpendicular to the field, or with the dipole anti-aligned with the field (i.e., the line from the $-q$ to the $+q$ charges opposite the direction of the field)?

In general, for any charge q, separation Δ, and constant electric field $\vec{E}_0$, what are (f) the net force and (g) net torque on the dipole?

Hint: To give a general answer for this problem, particularly part (g), you will want to make use of the unit coordinate vector $\hat{\imath} \equiv \hat{x} \equiv \hat{e}_0$.

3.2 Force on a dipole in a spatially varying electric field

Because the separation Δ for many types of dipoles tends to be very small (particularly for molecules), the electric field does not change much from the $-q$ to the $+q$ charges, and your answer for the torque in Problem 3.1 is a very good estimate even when the field isn't constant. Your answer for the force, though, only really tells you that the net force will be small. It doesn't really give an estimate of how big the force will be, so now we consider the force in the case of an electric field which isn't *exactly* constant.

Suppose that, like in the previous example, $q = 1$ nC and $\Delta = 0.01$ m, but now the electric field is spatially varying with the form

$$\vec{E}(x, y, z) = A\left\{((x+a)^2 - y^2)\,\hat{\imath} - 2(x+a)y\,\hat{\jmath}\right\},$$

where

$$\begin{aligned} A &= 30,000 \text{ V/m}^3 \\ a &= 1 \text{ m}. \end{aligned}$$

(**Note:** The units on A may seem strange at first, but you will see that the final answer for the electric field will always have the correct units of V/m.) What (a) will be the net force on the dipole?
Hints: Note that $y = 0$ for the two point charges, which will make your math a lot easier. Also, if you are doing things correctly, it will feel a lot like you are taking a derivative.

Now, (b) write down the result for a general electric field $\vec{E}(x, y, z)$. Then, using the definition of the derivative, explain why, for small Δ ($\Delta \to 0$), we can say

$$\vec{F}_{\text{tot}} = q\Delta \frac{\partial}{\partial x}\vec{E}(x = 0, y = 0, z = 0); \qquad (1)$$

i.e., we just have to take the partial x-derivative of $\vec{E}$ and evaluate the result at the location of the dipole.
Note: For those interested in advanced vector calculus, this simple general result is usually written

$$\vec{F}_{tot} = (\vec{p} \cdot \nabla)\vec{E}(\vec{x}),$$

where $\vec{p}$ is the dipole moment ($\equiv q\Delta\hat{\imath}$ in this case) and $\vec{x}$ is the location of the dipole. The force on a dipole thus is the derivative of the electric field at the location of the dipole along the direction of the dipole.

Finally, as a check, (c) apply your general formula, Eq. (3.4), to the specific case given in part (a).

3.3 Far-field of a dipole

In Problem 2, you explored the form of the electric field near the charges in an electric dipole. As we've mentioned, many dipoles are so tiny (such as molecules) that we generally are only interested in the electric field at locations $\vec{R} \equiv x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ very "far" from the dipole, by which we mean distances much larger than the separation between the charges ($|\vec{R}| >> |\vec{\Delta}|$). In this case, the field takes on a relatively simple mathematical form that is worth knowing.

To derive this form for yourself, (a) write down the net electric field $\vec{E}_{\text{d}}$ at location $\vec{R} \equiv x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ due to the two charges making up the dipole. Once again, your formula should remind you of some kind of derivative. Now, (b) explain how (for small $\Delta \to 0$) your formula gives

$$\vec{E}_{\text{d}} = -k_c q\Delta \frac{\partial}{\partial x}\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}}.$$

Hint: The tricky part is the sign. Think *carefully* about the definition of the derivative.

CHALLENGE: Finally, (c) using the above formula, take the derivative and massage it into its final, famous, form

$$\vec{E}_{\text{d}} = k_c \frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3}.$$

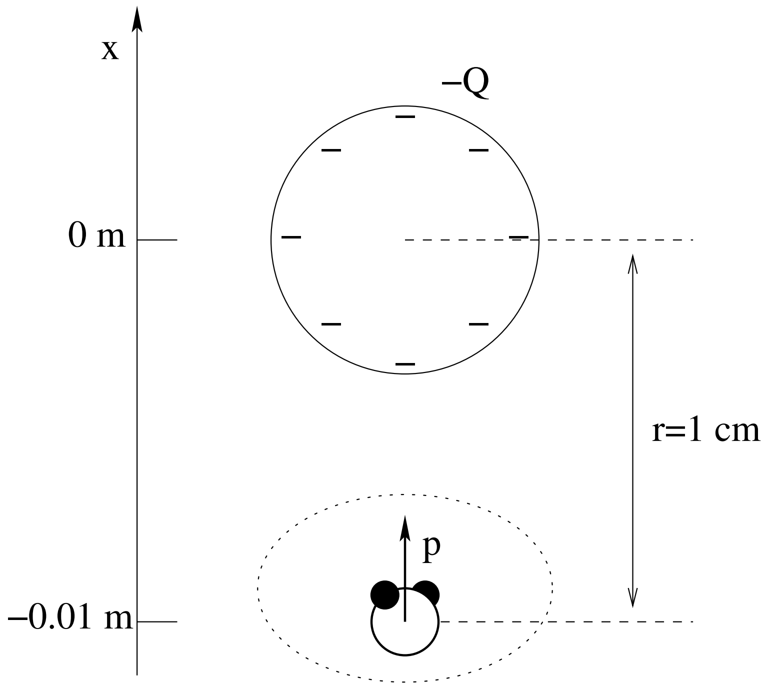

Figure 6: Electrostatic levitation of a water droplet below a metal sphere

3.4 Application: electrostatic levitation of water

As a potential application imagine the possibility of using the dipole moment $\vec{p}$ of the water molecule to levitate a droplet of water below a negatively charged metallic sphere. Figure 6 shows the geometry along with a schematic of one water molecule in an elliptically shaped droplet.

Due to the negative charge on the O^{2-} and the positive charges on the H^{+}'s, each water molecule in the droplet is electrically neutral but has a dipole moment of just about 6×10^{-30} C-m (coulomb-meters) running from the center of the (negative) oxygen ion and right through the center of the two H's. (It's a long story, but the two positive charges of the H's act just like a charge of +2 right between them.)

Now, because the sphere is negatively charged, the electric field at the location of a water molecule directly below it is in the upward direction. As we learned above, this means (under ideal circumstances where the molecules are all free to rotate to their preferred locations) that there is a torque acting on the water molecules that will eventually align their dipole moments $\vec{p}$ directly upwards toward the sphere.

If the droplet is to be able to levitate, the force on each of these dipoles due to the variations in the electric field from the charged sphere must balance the weight (mg) of each molecule. Using your previous result, Eq. (), for the force on each dipole due to the variations in the electric field (note that we've aligned the x axis with the dipole so your formula applies directly) and the fact that the electric field in this case has the value $\vec{E} = (k_c Q/x^2)\hat{\imath}$, (a) solve for the charge Q needed on the sphere to levitate the water molecules at a distance of 1 cm=0.01 m below the metal sphere.

Under these conditions, (b) is the strength of the electric field at the location of the water droplet less than the breakdown voltage for air (3×10^6 V/m)?

Note: Later in the course, we'll learn how to figure out how much voltage would be needed on the metal sphere to build up this much charge.

4 Integration to compute electric fields: the ribbon

Consider a ribbon of charge per unit area σ that lies in the xy plane of width $W = 2b$ and length $L = 2a$, extending from $x = -a$ to $x = +a$ and from $y = -b$ to $y = +b$ as in Figure 7, where we imaging a long ribbon so that $L >> W$.
Note: The width and length are written in terms of a and b in this way to reduce some of the messy algebra. *Hint:* You will want to work the problem through in terms of a and b and only at the end substitute back in terms for W and L.

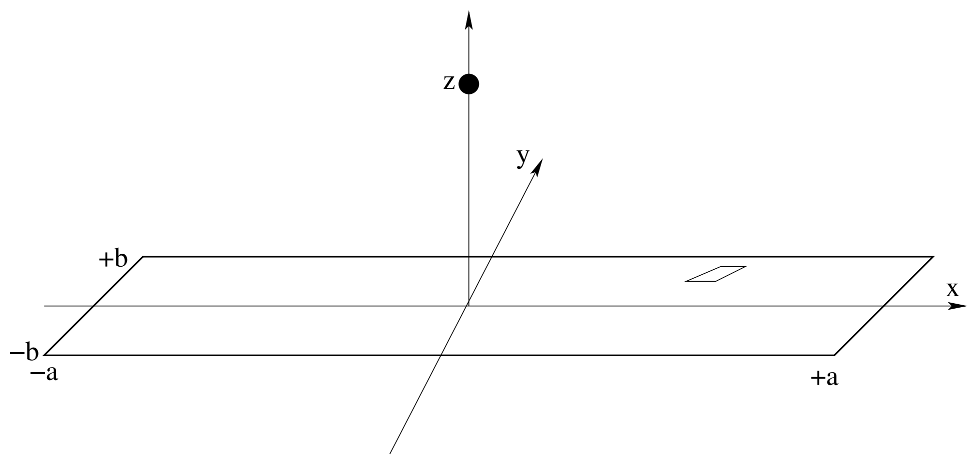

Figure 7: Ribbon of length $2a$ and width $2b$ in the xy plane

This problem asks you to use integration to compute the electric field at the location of the solid point in the figure, which is at a distance z above the exact center of the ribbon. You will then explore your answer in various limits ($z << W << L$, $W << z << L$, $W << L << z$) and see that you find familiar results.

4.1 Symmetry considerations

Explain briefly, in a sentence or two, how you know that the electric field at the point z has zero for its x- and y- components, and thus has only a z- component.

4.2 Field from a small patch

Consider a small patch of the ribbon, of length dx and width dy that extends from x to $x + dx$ and from y to $y + dy$. What is (a) the small amount of charge dq associated with this patch? Given that dx and dy are negligibly small, (b) what is the contribution of this patch to each of the components (E_x, E_y, E_z) of the total electric field at the point z, in terms of no quantities other than σ, x, y, z, dx, dy and k_c?

4.3 Integral for the electric field

Write down a double integral that gives the final result for E_z for the entire ribbon, $E_z = \iint \ldots \, dx \, dy$.
Note: Be sure to include the proper limits of integration.

4.4 x-integral

Using trig substitution, perform the x part of your double integral to show that

$$E_z = 2k_c\sigma za \int_{-b}^{+b} \frac{1}{(y^2+z^2)\sqrt{y^2+z^2+a^2}}\, dy.$$

Hint: The integral that you have to do described the field from the whole strip of the ribbon between y to $y+dy$ and $x=-a$ to $x=+a$ and so should work very much like computing the electric field from a line of charge, as done in lecture.

4.5 Final result

Using the fact that (Check it if you don't believe us!)

$$\frac{d}{dy}\left(\frac{1}{za}\tan^{-1}\frac{ya}{z\sqrt{y^2+z^2+a^2}}\right) = \frac{1}{(y^2+z^2)\sqrt{y^2+z^2+a^2}},$$

show that the final result for the electric field at the point z is

$$E_z = 4k_c\sigma \tan^{-1}\frac{ab}{z\sqrt{z^2+a^2+b^2}}. \tag{2}$$

4.6 Extracting results for the infinite line and infinite plane

In the limit that both $a \to \infty$ and $b \to \infty$, we clearly have an infinite sheet of charge. To make the math simpler, set $b = a$ (make the sheet square) and let $a \to \infty$. Then, using the fact that $\tan^{-1}(+\infty) \to \pi/2$, (a) derive the result for the field from an infinite sheet of charge, $E_z = 2\pi k_c\sigma$, independent of the value of z!

In the limit that $b \to 0$ and $a \to \infty$, we clearly have an infinite line of charge. Using the fact (the so-called small angle approximation) that $\tan^{-1} w \to w$ for small w, (b) derive the result for the infinite line of charge $E_z = 2k_c\lambda/z$, where λ is the total charge of the line per unit length.

4.7 Anticipated limiting behaviors

Now imagine that the point z is so far away that $z >> a$ and $z >> b$. From the perspective of someone sitting at that location, (a) would the charge ribbon more resemble a small point, a thin line, or a large sheet?

Now imagine that the point z is at an intermediate distance, where still $z >> b$, but now $z << a$ so that the right and left ends of the ribbon are very far away stretching almost out to the "horizon" from the perspective of someone sitting at the point z. (Remember the ribbon is long so that $a >> b$, which makes this kind of situation possible.) The this vantage point, (b) would the charge ribbon more resemble a small point, a thin line, or a large sheet?

Finally, imagine the situation where the point z is very close to the ribbon ($z << a$ and $z << b$). From the perspective of someone sitting on this point, (c) does the charge ribbon more resemble a small point, a thin line, or a large sheet?

4.8 Formulas for anticipated behaviors

Suppose you are in a situation where the electric field from the ribbon acts like a point charge. In terms of a, b and σ, (a) what would be the total charge q associated with the ribbon and what is the formula for electric field E_z expected at a distance z from such a charge in terms of k_c, σ, a, b and z.

Suppose now that the ribbon is acting like a thin line stretched along the x axis from $-a$ to a. In terms of b and σ, (b) what would be the charge put unit length λ of this thin line, and what is the formula you would expect for the electric field E_z at a distance z from an infinite line of charge with this charge per unit length? (Express your answer in terms of only k_c, σ, a, b and z.)

Finally, for cases where the ribbon is acting like an infinite sheet of charge, (c) what is the formula you would expect for the electric field E_z at a distance z in terms of no quantities other than k_c, σ, a, b and z. (Note that you probably won't need all these variables for your answer.)

4.9 Final verification of your answer

To confirm your expectations above, make a log-log plot (and attach it to your problem set) of the electric field E_z of the ribbon (Problem 4.5) versus z for the following specific values of the parameters: $a = 1000$ m, $b = 1$ m, $\sigma = 1$ nC/m^2. Also be sure to include on your plot the point, line, and plane behaviors from Problem 4.8 for comparison. Your plot should cover distances from $z = 10^{-3}$ m to $z = 10^6$ m.
Hint: If you are unfamiliar with log-log plots, you can simply plot $\log E_z$ versus $\log z$.