Physics 2213: Electromagnetism, Fall 2012

Problem Set # 10

(Due Friday, November 16 at 5:00pm *sharp.*)

Agenda and readings

Readings marked YF refer to sections from the text book, *University Physics*, 13th edition, volume 2, by Young and Freedman.

- Lec 21 11/06: Ampere's Law (w/ Displacement Current): 28.6, 29.7
- Lec 22 11/08: Apps of Ampere's Law: 28.7
- Lec 23 11/13: Motional EMF, Lenz's Law: 29.4, 29.3
- Lec 24 11/15: Faraday's Law, LR circuit (I): 29.1-2,30.1-2,30.4

Contents

1 Fields from moving charges: Biot-Savart approach

1.1 From a single moving charge

Do Exercise 28.1 from YF.

1.2 From a wire with a funny shape

Do Exercise 28.36 from YF.

2 Ampere's law: fundamentals

2.1 Basics

Do Exercise 28.42 from YF.

2.2 Co-axial cable

Do Exercise 28.45 from YF.

2.3 Solenoid

(a) In preparation to Exercise 28.48 from YF, sketch an infinite solenoid of n turns per unit length, and show the two Amperian loops you would use to find the magnitude B of the field at any point inside the solenoid and at any point outside the solenoid.

(b) Apply Ampere's law to your two loops from (a) to show that the magnitude of the field inside the solenoid is $B = \mu_0 n I$ and that the field at any point outside the solenoid is $B = 0$.

Do Exercise 28.48 from YF.

2.4 Toroidal solenoid

Apply Ampere's law to do Exercise 28.50 from YF. Be sure to include a sketch of your Amperian loop.

3 Vector calculus identities

In lecture we used a few vector identities. Proving these is good practice to review the basic operations of divergence, gradient, curl and cross-products.

3.1 Curl of a gradient

Compute $\nabla \times (\nabla f(x,y,z))$ for any function $f(x,y,z)$ and show that always

$$\nabla \times (\nabla f(\vec{r})) = 0.$$

3.2 Divergence of a cross product

(a) For any two vectors, $\vec{A}$ and $\vec{B}$, compute $\vec{C} = \vec{A} \times \vec{B}$.

(b) Compute $\nabla\times\vec{A}$ and $\nabla\times\vec{B}$ in terms of partial derivatives, getting a result like $\nabla\times\vec{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\hat{\imath} + \ldots$. (For $\vec{C}$ you can just copy your answer for $\vec{B}$ because you know it has to follow the same pattern.)

(c) Next, using your result for (a), write down $\nabla \cdot \vec{C}$, which will give you a series of terms,

$$\begin{aligned} \nabla \cdot \vec{C} &= \frac{\partial}{\partial x}\left(A_y B_z - A_z B_y\right) + \ldots \\ &= \left(\frac{\partial A_y}{\partial x} B_z + A_y \frac{\partial B_z}{\partial x} - \frac{\partial A_z}{\partial x} B_y - A_z \frac{\partial B_y}{\partial x}\right) + \ldots. \end{aligned}$$

(d) Then, collect all the terms that have pure A_x's (with no derivatives), pure A_y's, pure A_z's, pure B_x's, pure B_y's, pure B_z's, and factor out the pure A_x, A_y, A_z, B_x, B_y, and B_z's, respectively, producing an expression like

$$\nabla \cdot \vec{C} = A_x \left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}\right) + \ldots.$$

(e) Finally, using your results from (b), verify that your answer to (d) corresponds to the vector identity

$$\nabla \cdot \left(\vec{A} \times \vec{B}\right) = \left(\nabla \times \vec{A}\right) \cdot \vec{B} - \vec{A} \cdot \left(\nabla \times \vec{B}\right).$$

4 Inside a wire

Consider a long infinite wire with a finite, circular cross-section of radius R. Suppose that the wire carries a current with a *constant* current density of magnitude J spread evenly across the cross-section of the wire, as in Figure 1.

4.1 Magnetic field

(a) Make a sketch of the magnetic field lines inside the wire.

(b) Use Ampere's law to compute the magnitude of the magnetic field at any radius $r < R$ inside the wire.

4.2 Magnetic force

Supposing the charge carriers to be positive in this case, (a) sketch on the diagram above the direction of the force on the particles due to the magnetic field.

Do you expect the center of the wire to become positively or negatively charged in this case?

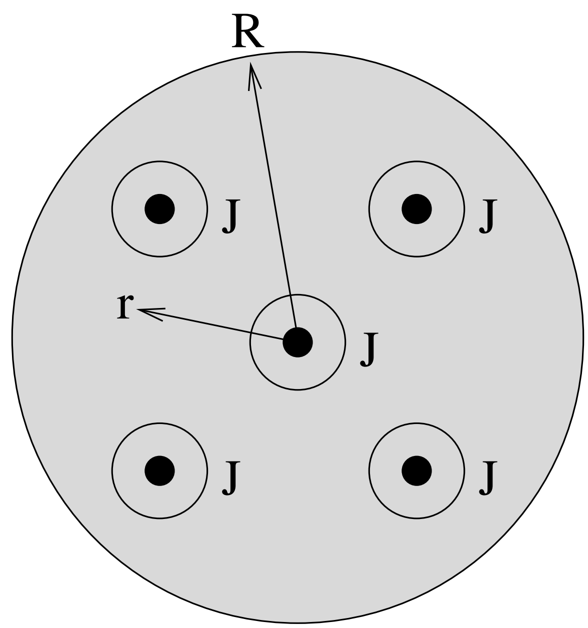

Figure 1: Cross-section (head on view) of infinite, circular wire, with current density $\vec{J}$ coming out of the page.

4.3 Equilibrium charge distribution

Once the center of the wire charges, equilibrium will ensure that there is no net force on any moving charges. (a) What must the electric field $\vec{E}(\vec{r})$ be to ensure equilibrium?

(b) What charge density $\rho(\vec{r})$ is required inside the wire to generate this electric field?
Hint: Don't forget Gauss's law, $\nabla \cdot \vec{E}(\vec{r}) = \rho(\vec{r})/\epsilon_0$!

4.4 Challenge: general results

(a) Using the fact that, in equilibrium, the total force $\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$ must be zero, and using the vector identity you proved in Problem 3.2, show that the charge density in the cross-section of the wire always obeys

$$\rho(\vec{r}) = \mu_0 \epsilon_0 \vec{v} \cdot \vec{J}(\vec{r}),$$

where $\vec{v}$ is the velocity of the charge carriers making up the current.

(b) By considering a length L of the wire (so that its total volume is LA), and using the fact that the current density $\vec{J}(\vec{r})$ and $\vec{v}$ point in the same direction, use your result in (a) to show that the charge per unit length of the wire is always

$$\lambda = \mu_0 \epsilon_0 v I,$$

where v is the speed of the carriers and I the total current in the wire.

(c) Finally, suppose that there are two identical such wires running parallel at a distance r from each other and which are long enough $L >> r$ that you may use the field formulas for infinite wires. The electric force between them will be repulsive and the magnetic force will be attractive. Show that the ratio of the magnitude of the electric force to the magnetic force between the wires is $\mu_0 \epsilon_0 v^2$.

(d) Given the above result, for what drift velocities v may we ignore this charging effect — so that, for instance, the electric force is less than 1/100 of the magnetic force?

5 Current sheets and energy in the magnetic field

5.1 Opposing parallel current sheets

Do Exercise 28.84 from YF to determine the magnetic field for two opposite current sheets.

Hint: Recall that we derived in lecture the result that, for a single current sheet, the magnitude of the magnetic field is $B = \frac{1}{2}\mu_0 n I$, where there are n wires per unit width of the sheet, each carrying a current I. Also, recall that the principle of superposition works for magnetic fields too.

5.2 Force between opposing current sheets

Take the length of the wires in both current sheets to be L, the total width of each sheet to be W and the distance between the sheets to be H.

(a) What is the total force F_{wire} on each wire in the upper sheet due to the the magnetic field from the lower sheet? Express your result in terms of nothing other than the given quantities n, I, L, W, H, and fundamental constants (ϵ_0, μ_0, 2, π, etc.).
Hint: Careful with factors of two!

(b) What is the total force on the upper sheet F_{sheet} due to the lower sheet?

(c) How much work W would be required to squeeze the plates together, bringing their separation H to zero.

(d) Compute $\iiint \frac{1}{2\mu_0} B^2 \, dV$ and show that it matches your answer in (c). This means that the energy density associated with the magnetic field (in this case, at least) is $B^2/(2\mu_0)$, a result very similar to what we found for the electric field.
Note: While the magnitudes work out, if you think very carefully about the signs of the various energy terms in this argument, something is a little bit off. The resolution of this conundrum is that there is an extra source of energy in the problem, namely the emfs (which could be batteries, for instance) that keep the currents going.