CORNELL UNIVERSITY

Department of Physics

Physics 2213 November 6, 2012

Prelim II, Fall 2012

NAME: Key

SECTION:

- To receive credit, you <u>must</u> place your answers in the boxes.
- Closed book; no notes; scientific, *non-graphing* calculators allowed.
- You may use the back of each page for additional scrap space if needed.
- **Important note #1:** The challenge problems are significantly more difficult. Give them a try when you get to them. If you know what to do right away, **go ahead**. If you get stuck on one, **move on** to the rest of the exam before spending much time on them.
- **Important note #2:** Successive parts within a given problem become more difficult. If you get stuck, skip to the next parts or problems and come back later.
- **Important note #3:** Sometimes we give you the answers for earlier parts of a problem. ("Show that ....") If you get stuck on one of these parts, skip ahead and use the given information to complete the later parts of the problem.

Problem	Score	Grader
1. (25 pts)		
2. (10 pts)		
3. (15 pts)		
4. (20 pts)		
5. (30 pts)		
Total (100 pts)		

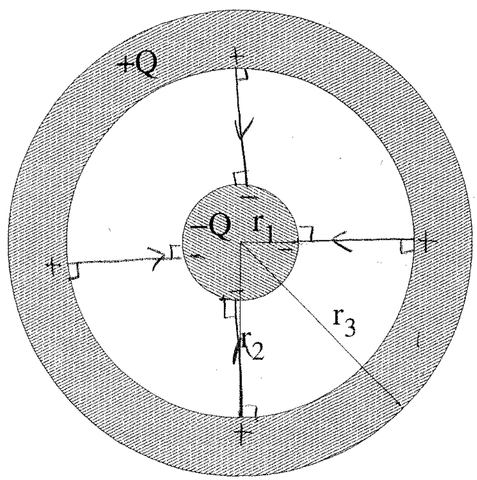
Contents

1 Spherical capacitor (25 pts *total*)

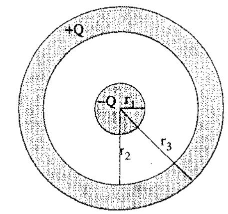
The figure below shows the cross-section of a hollow spherical conductor which inside contains another (solid) spherical conductor that sits centered with respect to the outer conductor. The outer conductor carries a charge $+Q$ and the inner conductor carries a charge $-Q$, and the charges have arranged themselves into their electrostatic equilibrium locations.

1.1 Field lines (6 pts)

Sketch the field lines associated with this situation, indicating the directions of the electric field lines clearly, and the signs of any charges at the beginning/ending points of the field lines.



1.2 Equilibrium charge distribution (3 pts)



Using the definition $\sigma \equiv Q/A$, determine the equilibrium surface charge density of each conductor surface ($r = r_1$, $r = r_2$ and $r = r_3$, respectively). Express all answers in terms of only Q, r_1, r_2, r_3 and any relevant mathematical (e.g., 2, π, etc.) and fundamental physical constants (ϵ_0, μ_0, etc.).

$$\sigma = \frac{Q}{A} = \frac{Q}{4\pi r^2}$$

$\sigma(r_1) = \boxed{\frac{-Q}{4\pi r_1^2}}$ $\sigma(r_2) = \boxed{\frac{+Q}{4\pi r_2^2}}$ $\sigma(r_3) = \boxed{0}$

1.3 Electric field near conducting surface (5 pts)

The electric field for this situation is $\vec{E}(\vec{r}) = \begin{cases} -\frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & r_1 < r < r_2 \\ 0 & \text{otherwise} \end{cases}$.

Show that your result from Problem 1.2 for $\sigma(r_1)$ is consistent with this electric field.

Metal surface

$$E = \sigma/\epsilon_0$$

$$E(r_1) = \left(\frac{-Q}{4\pi r_1^2}\right)/\epsilon_0$$

$$= \frac{-Q}{4\pi\epsilon_0 r_1^2}$$

1.4 Energy density and total energy stored (6 pts)

What is the energy density $u(r)$ stored in the electric field for points $r_1 < r < r_2$ in terms of Q, r, ϵ_0 and any relevant mathematical constants?

$$u(r) = \frac{\epsilon_0}{2}|E(r)|^2 = \frac{\epsilon_0}{2}\left(\frac{-Q}{4\pi\epsilon_0 r^2}\right)^2 = \frac{Q^2}{32\pi^2\epsilon_0 r^4}$$

$$u(r) = \boxed{\frac{Q^2}{32\pi^2\epsilon_0 r^4}}$$

Next, using this energy density, compute the total energy $U^{(tot)}$ stored in the electric field for this situation in terms of Q, r_1, r_2 and ϵ_0 and mathematical constants.
Hint: $\int_{r_1}^{r_2} \frac{dr}{r^2} = (r_1^{-1} - r_2^{-1})$.

$$U^{(tot)} = \int u(r)\,dV = \int_{r_1}^{r_2} u(r) \cdot \underbrace{4\pi r^2 dr}_{\text{volume of spherical shell } dV}$$

$$= \int_{r_1}^{r_2} \frac{Q^2}{\cancel{32\pi^2}\epsilon_0 r^4} \cdot \cancel{4\pi} r^2 dr \quad 8\pi$$

$$= \frac{Q^2}{8\pi\epsilon_0}\int_{r_1}^{r_2}\frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon_0}(r_1^{-1} - r_2^{-1})$$

$$U^{(tot)} = \boxed{\frac{Q^2}{8\pi\epsilon_0}(r_1^{-1} - r_2^{-1})}$$

1.5 Challenge: Capacitance (5 pts)

Use your result from Problem 1.4 to compute the capacitance C between the inner and outer spheres in terms of only r_1, r_2, ϵ_0 and mathematical constants.

$$U = \frac{Q^2}{2C} = \frac{Q^2}{8\pi\epsilon_0}(r_1^{-1} - r_2^{-1})$$

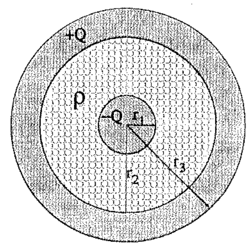
$$\Rightarrow \frac{1}{C} = \frac{1}{4\pi\epsilon_0}(r_1^{-1} - r_2^{-1})$$

$$\Rightarrow C = 4\pi\epsilon_0 (r_1^{-1} - r_2^{-1})^{-1}$$

$$= \frac{4\pi\epsilon_0}{\frac{1}{r_1} - \frac{1}{r_2}} = \frac{4\pi\epsilon_0 r_1 r_2}{r_2 - r_1}$$

$$C = \boxed{\frac{\epsilon_0 4\pi r_1 r_2}{r_2 - r_1}}$$

2 Spherical resistor (10 pts *total*)



Now consider the exact same situation from Problem 1 but with the space between the spheres filled with a conductive material with a resistivity ρ, as in the figure to the right.

Recall that the electric field in this case is

$$\vec{E}(\vec{r}) = \begin{cases} 0 & r < r_1,\ r > r_2 \\ -\frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & r_1 < r < r_2 \end{cases}.$$

2.1 Current density and current (6 pts)

For both questions below, express your answers in terms of no quantities other than r_1, r_2, Q and ρ, and fundamental physical and mathematical constants (ϵ_0, μ_0, 2, π, etc.).

What is $J(r_1)$, the magnitude of the current density passing through the conductive material *just outside* the inner sphere?

$$J = \sigma|E| = \frac{1}{\rho}|E| = \left|\frac{-Q}{4\pi\epsilon_0\rho r^2}\right|$$

$$\Rightarrow J(r_1) = \frac{Q}{4\pi\epsilon_0\rho r_1^2}$$

$J(r_1) = \boxed{\frac{Q}{4\pi\epsilon_0\rho r_1^2}}$

What, then, is the total current I flowing from the outer to the inner sphere?

$$I = J \cdot A = J(r_1)\cdot 4\pi r_1^2 = \frac{Q}{\cancel{4\pi}\epsilon_0\rho \cancel{r_1^2}}\cdot \cancel{4\pi r_1^2}$$

$I = \boxed{\frac{Q}{\epsilon_0\rho}}$

2.2 Challenge: Resistance (4 pts)

What is the resistance R between the outer and inner spheres? Express your answer in terms of no quantities other than r_1, r_2, ρ, and mathematical constants (2, π, etc.)

Hint: There are several ways to do this problem. By far, the quickest uses the capacitance C from the previous problem. If you don't see how to do it this way, you may wish to come back to this question later.

$$V = IR$$

$$\Rightarrow R = \frac{V}{I}$$

$$CV = Q \Rightarrow V = \frac{Q}{C}$$

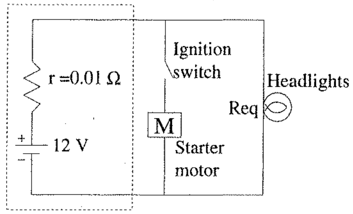
$$\Rightarrow R = \frac{V}{I} = \frac{Q/C}{I} = \frac{Q}{IC}$$

$$= \frac{Q \rightarrow I}{\left(\frac{Q}{\epsilon_0\rho}\right)\frac{\epsilon_0 4\pi r_1 r_2}{r_2 - r_1}} = \frac{\rho(r_2 - r_1)}{4\pi r_1 r_2}$$

$R = \boxed{\frac{\rho(r_2 - r_1)}{4\pi r_1 r_2}}$

3 Automobile circuits and power (15 pts *total*)

The schematic to the right shows a simplified circuit diagram for a car. On the left of the diagram, you have the 12 V battery with an internal resistance $r = 0.01\ \Omega$. The battery powers the headlights, and, when the ignition switch is turned to the "on" position, the battery also powers the starter motor.



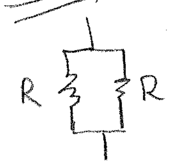
3.1 Headlight resistance (9 pts)

A car usually has two headlights, each with a resistance R, wired together **in parallel** as a unit with an equivalent resistance R_{eq}. A standard automotive headlight has a nominal power rating of 55 W. This means that when each light is wired *by itself* to an ideal emf (no internal resistance) of $\mathcal{E}$=12 V, the power consumed is 55 W.

Use the above facts to determine the resistance R of each headlight individually and the equivalent resistance R_{eq} of the two headlights taken together as a unit. Give numerical answers in units of ohms.

$$P = IV = \left(\frac{V}{R}\right)V = \frac{V^2}{R}$$

$$\Rightarrow R = \frac{V^2}{P} = \frac{(12V)^2}{55W} = 2.6\,\Omega$$



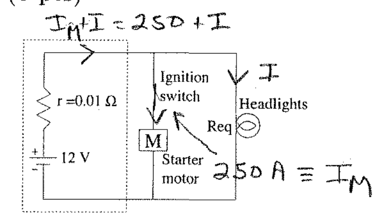
$$R_{eq}^{-1} = R^{-1} + R^{-1} = 2R^{-1} \Rightarrow R_{eq} = \tfrac{1}{2}R$$

$$\text{Or, } R_{eq} = \frac{R\cdot R}{R+R} = \frac{R^2}{2R} = \tfrac{1}{2}R$$

$R =$ $\boxed{2.6}$ Ω $\qquad$ $R_{eq} =$ $\boxed{1.3}$ Ω

3.2 Challenge: Starter motor current (6 pts)

Now, consider a car with a **special** high-power headlight system with an **an equivalent resistance** $R_{eq} = 1\ \Omega$. (This is **not** the answer to the previous question, just a very rough approximation to it.)



When you leave these headlights on while you start the car, you observe that the headlights dim while the starter motor is running. Suppose the starter motor under these conditions draws 250 A of current (a realistic number for a car with a large engine). By what fraction is the power to the headlights reduced? E.g., what is P/P_0, where P_0 is the power to the headlights when the ignition switch is in the "off" position, and P is the power to the lights when the ignition switch is in the "on" position.

For P_0

$$0 = +\mathcal{E} - I_0 r - I_0 R_{eq} \Rightarrow I_0 = \frac{\mathcal{E}}{r + R_{eq}}$$

$$\Rightarrow P_0 = I_0^2 R_{eq} = \mathcal{E}^2 \frac{R_{eq}}{(r+R_{eq})^2} = \frac{(12V)^2 \cdot 1\Omega}{(1.01\,\Omega)^2}$$

$$= 140W \quad (2 \text{ sig figs})$$

For P

$$0 = \mathcal{E} - (I_M + I)r - I R_{eq}$$

$$\Rightarrow I = \frac{\mathcal{E} - I_M r}{r + R_{eq}}, \quad P = I^2 R_{eq} = \left(\frac{\mathcal{E} - I_M r}{r + R_{eq}}\right)^2 R_{eq} = 89W$$

$$\Rightarrow \frac{P}{P_0} = \frac{I^2 R_{eq}}{I_0^2 R_{eq}} = \frac{I^2}{I_0^2} = \frac{\frac{(\mathcal{E} - I_M r)^2 \cancel{R_{eq}}}{\cancel{(r+R_{eq})^2}}}{\frac{\mathcal{E}^2 \cancel{R_{eq}}}{\cancel{(r+R_{eq})^2}}} = \left[\frac{\mathcal{E} - I_M r}{\mathcal{E}}\right]^2$$

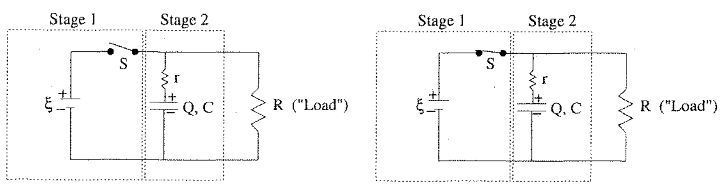
$\frac{P}{P_0} =$ $\boxed{0.63}$

(63%)

4 AC to DC conversion (20 pts *total*)

A "rectifier" circuit converts AC power (where the voltage is a sine wave oscillating from positive to negative and back again) to DC power (where the voltage is constant). Your laptop charger is a good example of such a circuit.

The circuit diagrams below show such a rectifier circuit connected to the device to be powered, the so-called "load" resistance R. The first stage (Stage 1) of a rectifier circuit acts like an ideal emf $\mathcal{E}$ connected to a switch S which is alternately open for a time Δt (Diagram (a)) and closed for a time Δt (Diagram (b)), in an infinitely repeating cycle. To make a more steady supply of power, a second stage (Stage 2), consisting of a capacitor C wired across Stage 1, is inserted in the circuit. Finally, to be realistic, we must include an internal resistance r for the capacitor.

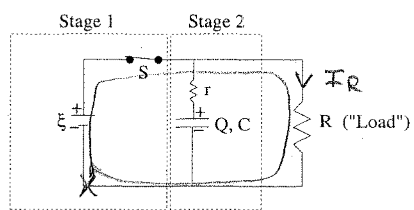


Rectifier circuit to provide steady DC power to load resistance R: (a) open circuit phase, (b) closed circuit phase. Switch S alternately remains open and closed for time intervals Δt.

4.1 Closed phase: current to the load (4 pts)

Using an appropriately chosen Kirchhoff loop, show that the current I_R through the load R obeys $I_R = \mathcal{E}/R$ for those times when the switch S is closed (as in (b) above).

To receive full credit, you must indicate on the figure to the right the loop you are using, and below you must provide the details (terms and correct signs) for the corresponding Kirchhoff equation.
(Additional workspace next page.)



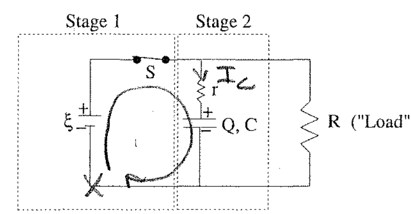
$O = + \mathcal{E} - I_R R$

$\Rightarrow I_R = \mathcal{E}/R$

4.2 Closed phase: capacitor charge (5 pts)

The time Δt is sufficiently long that the capacitor charges fully during the closed phase. Using an appropriately chosen Kirchhoff loop, show that the fully charged capacitor has a charge, $Q_F = C\mathcal{E}$.

To receive full credit, you must provide the details (terms and correct signs) for the corresponding Kirchhoff equation, and you must explain in a brief phrase or sentence any assumptions you make about any relevant currents.



$I_C = 0$ when fully charged $\frac{dQ}{dt} = 0 = I$ b/c Q is const.

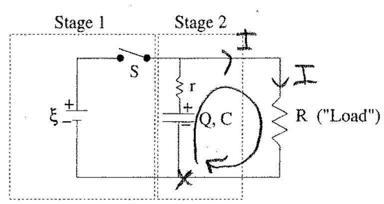
$CV = Q$

$O = + \mathcal{E} - \cancel{I_C}^{0} r - \frac{Q}{C}$

$\Rightarrow Q = \mathcal{E} C$

4.3 Open phase (6 pts)

Suppose that the capacitor is fully charged with the charge Q_F from above, and that, at time $t = 0$, the switch opens as in the figure to the right.



What will be

- $I_R(0^+)$ (the value of the current through the load R *right after* the switch opens at $t = 0$),
- $I_R(\infty)$ (the final value of the current through the load as $t \to \infty$),
- and τ (the time constant characterizing the exponential approach to the final value)?

Express your results in terms of only $\mathcal{E}$, R, r, and C.

@ $t = 0^+$ $\quad Q = \mathcal{E}C$ from 4.2

$$0 = +\frac{Q}{C} - Ir - IR \Rightarrow I = \frac{\frac{Q}{C}}{r+R} = \frac{\frac{\mathcal{E}C}{C}}{r+R} = \frac{\mathcal{E}}{r+R}$$

@ $t \to \infty$

$$Q = \text{const} \Rightarrow -\frac{dQ}{dt} = I = 0$$

Decay: $$0 = \frac{Q}{C} - Ir - IR \Rightarrow \frac{Q}{C} = I(r+R) = -\frac{dQ}{dt}(r+R)$$

$$\Rightarrow \frac{dQ}{dt} = \frac{Q}{(r+R)C}$$ ← looks like std cap eq. but with $(r+R)C$ instead of RC

$\Rightarrow \tau = (r+R)C$

(Additional workspace on the next page.)

$I_R(0^+) = \boxed{\frac{\mathcal{E}}{r+R}}$

$I_R(\infty) = \boxed{0}$

$\tau = \boxed{(r+R)C}$

4.4 Challenge: Capacitor design (5 pts)

A typical laptop operates internally at $\mathcal{E}$=12 V DC and requires a power of 60 W, corresponding to a current of I_R=5 A and an equivalent load resistance of R=2.4 Ω.

Suppose that you power this laptop with standard US AC power, so that $\Delta t = 8.3\text{ ms} \equiv 0.0083$ s. Further suppose that you use a rectifier circuit with a capacitor $C = 75$ mF $\equiv 0.060$ F that has an internal resistance $r = 0.1$ Ω. Assuming that $\mathcal{E}$=12 V, show that this capacitance is large enough to ensure that the laptop current I_R at the end of the "open phase" (just before the the switch S closes again and the capacitor begins to recharge) does not drop below 4.5 A.

Decay is exp toward zero w/ τ

$$\Rightarrow I = I_0 e^{-t/\tau} = \frac{\mathcal{E}}{r+R} e^{-\frac{t}{(r+R)C}}$$

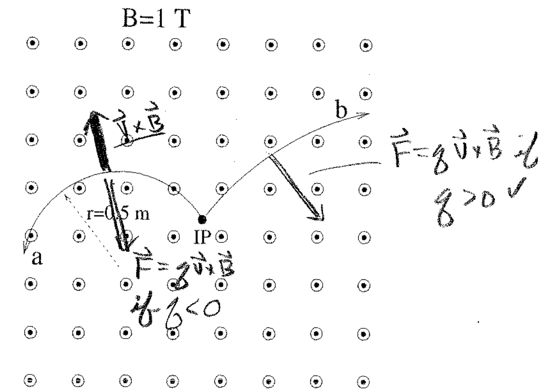
At $t = \Delta t$, we get

$$= \frac{\mathcal{E}}{r+R} e^{-\frac{\Delta t}{(r+R)C}} = 4.6\text{A} > 4.5\text{A} ✓$$

5 Forces on particles and wires in magnetic fields (30 pts *total*)

5.1 Particle discovery experiment (8 pts)

The figure below shows the results of the production of two elementary particles (e.g., electrons, positrons, mesons, etc.) in an experiment. The particles were produced at the interaction point (IP) in the diagram and then followed the curved trajectories shown in the figure. (Antimatter actually was discovered by particles curving the *opposite way* from what was expected in an experiment just like this!)



Paths of two elementary particles, (a) and (b), produced at interaction point IP in a region of constant magnetic field $B = 1$ T pointing out of the page.

From the trajectories in the figure, <u>determine</u> the sign of each charge and <u>circle</u> the correct choices below.

Sign of charge on Particle (a) = $+$ / $-$.

Sign of charge on Particle (b) = $+$ / $-$.

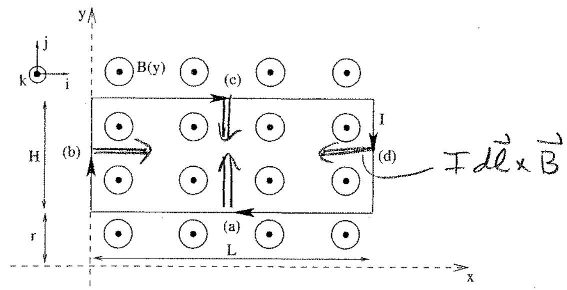
5.2 Forces on a wire loop (8 pts)

The rest of this exam (Problems 5.2-5.4) concerns the situation in the figure below.

In this scenario, a rectangular loop of length L and height H sits at a distance r above the x-axis in the xy-plane as indicated in the figure. The loop carries a current I in the clockwise direction and experiences a magnetic field in the $\hat{k}$ direction that depends only on the y coordinate according to the formulas

$$\begin{aligned} \vec{B} &= B(y)\,\hat{k} \\ B(y) &= B_0\, e^{-ay}, \end{aligned}$$

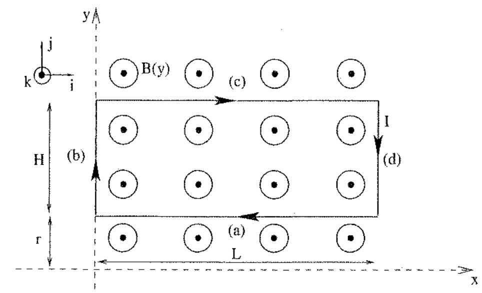
where B_0 and a are some given *positive* constants. Note that, because this field depends only on y, the field, for any given value of y, is completely constant and does not depend on x at all. Finally, because $a > 0$, this field **gets smaller with distance** from the x-axis for points in the upper-half of the xy-plane.



Rectangular current-carrying wire loop in a magnetic field.

Sketch, **directly on the figure above**, the directions of the forces ($\vec{F}_a$, $\vec{F}_b$, $\vec{F}_c$, $\vec{F}_d$) which the field $\vec{B}$ exerts on **each of the four sides** (a), (b), (c), (d) that make up the rectangular wire loop, respectively.

(Exam continues next page.)



$\vec{B} = B(y)\,\hat{k} \quad B(y) = B_0 e^{-ay}$

5.3 Relative magnitudes (7 pts)

Circle the option (1-4) below which correctly represents the relative magnitudes of the forces from the magnetic field on each of the sides.

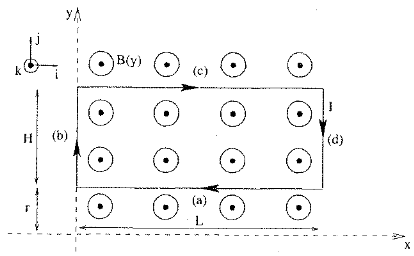
(1) $|\vec{F}_a| = |\vec{F}_c|$ and $|\vec{F}_b| > |\vec{F}_d|$ (2) $|\vec{F}_a| = |\vec{F}_c|$ and $|\vec{F}_b| < |\vec{F}_d|$

(3) $|\vec{F}_a| < |\vec{F}_c|$ and $|\vec{F}_b| = |\vec{F}_d|$ (4) $|\vec{F}_a| > |\vec{F}_c|$ and $|\vec{F}_b| = |\vec{F}_d|$

B is larger for (a) b/c a is closer + field is exp decaying w/ y!

same fields + |dl| just opp direction

5.4 Net force on loop (7 pts)



$\vec{B} = B(y)\,\hat{k} \quad B(y) = B_0 e^{-ay}$

What is the **total net force** on the wire loop due to the magnetic field $\vec{B}$?
Express your answer in terms of no quantities other than the given physical quantities r, L, H, I, B_0, a, and fundamental physical and mathematical constants (e.g., ϵ_0, μ_0, 2, π etc.).
Be sure to give both magnitude and direction.

(b) + (d) cancel.

$\vec{F}_a = I\vec{L}\times\vec{B}$ (since $\vec{B}$ is constant along (a))

$= ILB(r)(\sin 90)(+\hat{j}) = ILB_0 e^{-ar}\hat{j}$

$\vec{F}_b = I\vec{L}\times\vec{B}$

$= ILB(r+H)(\sin 90)(-\hat{j}) = -ILB_0 e^{-a(r+H)}\hat{j}$

$\vec{F}_{tot} = \vec{F}_a + \vec{F}_b = ILB_0\left(e^{-ar} - e^{-a(r+H)}\right)\hat{j}$

$\vec{F}_{tot} = \boxed{ILB_0 e^{-ar}\left(1-e^{-aH}\right)\hat{j}}$

End of exam.