Physics 211 Sections 1 & 70, fall 2012

Note on constant acceleration motion
09/11/12

Derivation of kinematic equations for constant acceleration.

The kinematic equations for constant acceleration come from the definitions of average velocity and acceleration, plus one observation about average velocity when acceleration is constant.

The definitions of average acceleration and average velocity:

$$\bar{a} = \frac{\Delta v}{\Delta t}$$

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

Solving these equations for Δx and Δv gives

$$\Delta v = \bar{a}\,\Delta t \qquad (1)$$

$$\Delta x = \bar{v}\Delta t \qquad (2)$$

The book uses the notation that the change in time is

$$\Delta t = t$$

i.e. the initial time is zero, so $\Delta t = t_f - t_i = t - 0 = t$. Similarly,

$$\Delta v = v - v_0$$

$$\Delta x = x - x_0$$

Substituting this notation into equations (1) and (2) above gives

$$v - v_0 = \bar{a}\,t \qquad (3)$$

$$x - x_0 = \bar{v}\,t \qquad (4)$$

If acceleration is a constant value a, then the average acceleration is just the constant acceleration: $\bar{a} = a$.

For constant acceleration, the velocity increases at a steady rate, so between the initial time ($t = 0$) and the final time t, the average velocity is just the geometric average (*note this is only true for constant acceleration!):*

$$\bar{v} = \frac{v_0+v}{2} \qquad (5)$$

Now, solving equation (3) for v with $\bar{a} = a$ gives one kinematic equation for constant acceleration:

$$v = v_0 + a\,t \qquad (K1)$$

Substituting equation (K1) into equation (5) gives

$$\bar{v} = \frac{1}{2} v_0 + \frac{1}{2} (v_0 + a\,t)$$
$$= \frac{1}{2} v_0 + \frac{1}{2} v_0 + \frac{1}{2} a\,t$$
$$\Rightarrow$$
$$\bar{v} = v_0 + \frac{1}{2} a\,t \quad (6)$$

Substituting equation (6) into equation (4) and solving for x gives

$$x - x_0 = \left(v_0 + \frac{1}{2} a\,t\right) t = v_0\,t + \frac{1}{2} a\,t^2$$

$$\Rightarrow$$

$$x = x_0 + v_0\,t + \frac{1}{2} a\,t^2. \quad \text{(K2)}$$

That’s our second kinematic equation. Next, substitute equation (5) into equation (4) and solve for x to get

$$x - x_0 = \bar{v}\,t = \left(\frac{v+v_0}{2}\right) t$$
$$\Rightarrow$$
$$x = x_0 + \frac{1}{2} (v + v_0)\,t \qquad \text{(K3)}$$

There’s one more: to get the last kinematic equation, square both sides of (K1)

$$v^2 = {v_0}^2 + a^2\,t^2 + 2\,a\,v_0\,t \qquad (7)$$

and multiply (K2) by $2\,a$

$$2\,a\,x = 2\,a\,x_0 + 2\,a\,v_0\,t + a^2\,t^2$$

and solve for $\left(2\,a\,v_0\,t + a^2\,t^2\right)$ to get

$$2\,a\,v_0\,t + a^2\,t^2 = 2\,a\,(x - x_0) \qquad (8)$$

Substitute equation (8) into equation (7) to get

$$v^2 = {v_0}^2 + 2\,a\,(x - x_0) \quad \text{(K4)}.$$

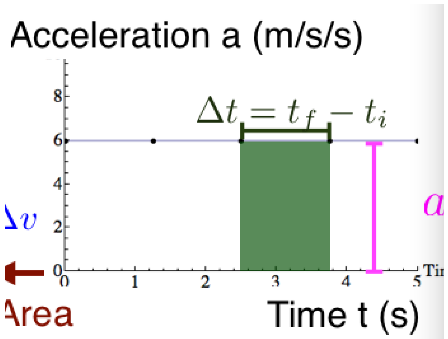
So to summarize, here are the kinematic equations for constant acceleration:

$$v = v_0 + a\,t \quad \text{(K1)}$$
$$x = x_0 + v_0\,t + \frac{1}{2} a\,t^2. \quad \text{(K2)}$$
$$x = x_0 + \frac{1}{2} (v + v_0)\,t \qquad \text{(K3)}$$
$$v^2 = {v_0}^2 + 2\,a\,(x - x_0) \quad \text{(K4)}$$

■ Plots of motion

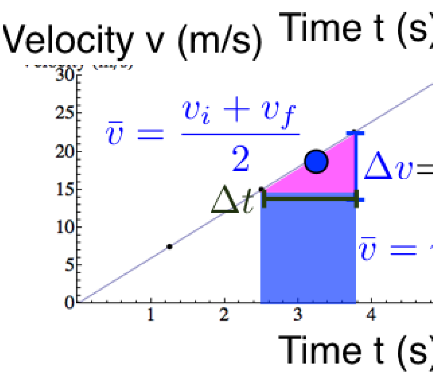
Let's think about (K1) and (K2); the others follow from these [as shown above explicitly for equation (K4)]. What do these equations mean intuitively? It turns out you can understand the terms in this equation in terms of areas:

For constant acceleration, the acceleration vs. time is flat. Equation (K1) is saying that the change in velocity, $\Delta v = v - v_0$, in a time Δt, is the area under the acceleration vs. time curve:



$v - v_0 = \Delta v = a\,t \;=\; a\,\Delta t.$

Simularly, in Eq. (K2), the change in position, $x - x_0 = \Delta x$, is an area under the velocity vs. time curve:



In time interval Δt, the area under the velocity curve can be found in two parts:

the blue rectangle has height v_0 and width Δt, so its area is $A_{\text{blue\_rect}} = v_0\,\Delta t$. The pink trinagle has base Δt and height Δv, so its area is $A_{\text{pink\_tri}} = \frac{1}{2}\,\Delta t\,\Delta v$. So the total area under the curve is (substituting $\Delta t = t - 0 = t$)

$A = A_{\text{blue\_rect}} + A_{\text{pink\_tri}} = v_0\,t \;+\; \frac{1}{2}\,\Delta v\,t.$

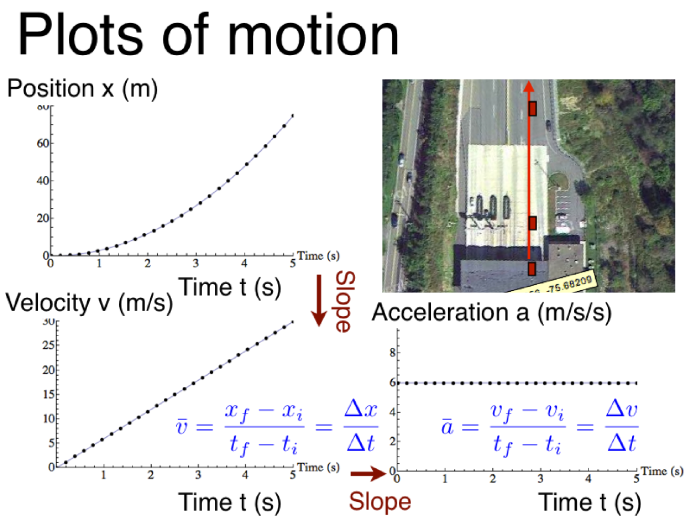
Now, we know that $\Delta v = a\,\Delta t = a(t - 0) = a\,t$ from the previous figure; substituting this in gives

$A = v_0\,t + \frac{1}{2}\,(a\,t)\,t = v_0\,t + \frac{1}{2}\,a\,t^2$

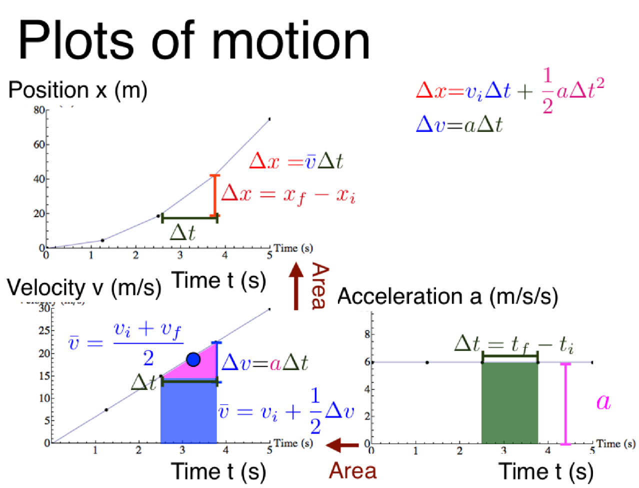
This is just $x - x_0$ from (K2) :

$x - x_0 = v_0\,t + \frac{1}{2}\,a\,t^2.$

Now, the velocity vs time curve is the slope of the position vs time curve, and the acceleration vs time curve is the slope of the velocity vs time curve, as shown on this slide in class:



Similarly, to get change in velocity from acceleration, you take the area under the curve, and similarly for position from velocity, as shown on this slide from class:



Although I illustrated these relations for constant acceleration, they are true for any motion.

This class does not use calculus, but if you were to take a calculus course, you would find that the operations of finding the slope (called the “derivative” in calculus class) and computing areas under curves (called “definite integrals” in calculus class) are *inverse operations* of each other. But for this class, I just want you to know visually the relation between position, velocity, and acceleration plots—especially for the constant acceleration case discussed in this note.