CS 70 Discrete Mathematics and Probability Theory

Fall 2012 Vazirani

Mock Final Fall 2012

PRINT your name: ______________________, ______________________
(last) (first)

PRINT your login: cs70-_____

CIRCLE the first letter of your login: a b c d e f g h i j k l m n o p q r s t u v w x y z

CIRCLE the second letter of your login: a b c d e f g h i j k l m n o p q r s t u v w x y z

CIRCLE your GSI name and section time: Lisha 10-11 Andrew 10-11 Andrew 11-12

Ian 12-1 Ian 1-2 Rahul 12-1 Baruch 1-2 Baruch 3-4

Shivaram 2-3 Shivaram 6-7 Jiung 4-5 Jiung 5-6

PRINT your GSI name and section time: ______________________

Name of the person sitting to your left: ______________________

Name of the person sitting to your right: ______________________

You may consult three single-sided sheets of notes. You may write your answers in the form of factorials, powers or binomial coefficients (n choose k), unless explicitly told otherwise. Calculators are not permitted, and cellphones must be turned off and put away. Do all your work on the pages of this examination. Give reasons for all your answers. Good Luck!

Do not turn this page until your instructor tells you to do so.

Problem 1		Problem 5	
Problem 2		Problem 6	
Problem 3		Problem 7	
Problem 4		Problem 8	
Total			

1. [10 pts] **Russian Multiplication, Fall 2000**

The so-called *Russian algorithm* for multiplying two positive integers a, b is defined recursively as follows:

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algorithm mult(a, b)
  if a = 1 then return(b)
  else return(mult(⌊a/2⌋, 2b) + b × (a mod 2))
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Your task is to prove, using (strong) induction on the first input a, that the program correctly outputs the product ab.

2pts (a) Write down the statement $P(a)$ that has to be proved by induction.

2pts (b) Prove the base case $P(1)$.

6pts (c) Prove $P(a)$ for all $a > 0$ by induction.

2. [10 pts] **Fascinating Rhythm, Spring 2006** There is no clock in my kitchen. However, the faucet drips every 12 seconds and the toaster oven beeps every 7 seconds after I turn it on. I'd like to boil an egg for exactly three minutes (180 seconds). My plan is to turn on the oven exactly when the faucet drips, and then start boiling the egg when the faucet drips for the D-th time, and stop when the toaster beeps for the B-th time.

5pts (a) Write an equation involving the variables D, B, and constants 12, 7 and 180 that must be satisfied for this plan to work.

5pts (b) Calculate values B and D that satisfy your equation. What algorithm would you use in general to solve this problem?

3. [10 pts] Event Modulus, Spring 2006 Answer each of the following and give a one or two line justification:

(a) Let X be uniformly distributed on $\{0, 1, \ldots, 100\}$. What is $\Pr[2X \equiv 21 \bmod 101]$?

(b) Let X be uniformly distributed on $\{0, 1, \ldots, 99\}$. What is $\Pr[2X \equiv 21 \bmod 100]$?

(c) Let X be uniformly distributed on $\{0, 1, \ldots, 36\}$. What is $\Pr[X^2 \equiv 36 \bmod 37]$?

4. [15 pts] They're Breaking Up, from "Fifty Challenging Problems in Probability"
A bar is broken at random in two places. Find the expected lengths of the left-most piece, the middle piece, and the right-most piece.

5. [10 pts] Cereal Collector, Fall 2006 Each cereal box has a baseball card. Suppose that there are 100 distinct cards, and each cereal box contains a random card. You consume a cereal box each week. At the end of the year, (assuming there are exactly 52 weeks in the year), what is the probability that you have collected one particular card, say, Barry Bonds? An expression involving powers, factorials, or binomial coefficients is sufficient. No need to calculate the precise number.

What is the expected number of distinct baseball cards you would have collected?

6. [15 pts] Double Decker, inspired by "Fifty Challenging Problems in Probability" Consider two standard, shuffled decks of cards (i.e. 52 distinct cards). Lay out cards of the first deck on a table, one by one. Lay the cards of the second deck, one by one, beneath the first deck. A match occurs when the ith card in the top row matches the ith card in the bottom row.

(a) What is the probability that there is a match in the first place, i.e. the first card from the top row matches the first card from the second row?

(b) What is the probability that the first and last cards match, i.e. there is a match at position $i = 1$ and at position $j = 52$.

(c) What is the expected number of matches in this experiment?

(d) What is the variance of the total number of matches in this experiment?

7. [10 pts] Arithmetic/polynomials true or false

For each of the following statements, say whether the statement is true or false. You need not provide any justification for your answer; however, you may be awarded partial credit for an incorrect answer if you attempt a justification.

2pts (a) If p is a prime, then for any other prime $q > p$, there do not exist integers $a, b \neq 1$ such that $ab = p \bmod q$.

2pts (b) If n is not a prime and $a > 0$, then it is always the case that $a^n \neq a \bmod n$.

2pts (c) If we take all powers of 3 mod 55, then we get a permutation of the numbers $1, 2, \ldots, 54$.

2pts (d) In the field modulo 29, there is exactly one polynomial of degree 5 that passes through 6 given points.

2pts (e) In the field modulo 29, there are exactly 29 polynomials of degree 5 that pass through 5 given points.

2pts (f) Using the Berlekamp/Welsh coding, we can recover lost digits as long as not more than one quarter of the digits are lost.

8. [20 pts] Monty Hall Variant, Spring 2008 Tired of hosting the same game year after year, Monty Hall decided to make some changes to his game. There are still three doors, but now one contains 1000 dollars, one contains 500 dollars, and one contains 0 dollars, with the order of the prizes randomly permuted. The contestant first selects a door. Then she has the choice of paying X dollars for Monty to open, among the two unchosen doors, the one that contains the smaller amount of money. If the contestant paid Monty, she then has the choice of switching to the other unopened door.

(a) Suppose the contestant refuses to pay Monty. In this case, what is the expected value of her prize?

(b) Suppose the contestant decides to pay, and then Monty opens a door that contains $500. Given this, what is the expected value of her prize if she switches?

Given this, what is the expected value of her prize if she sticks with her original door?

Multiple Choice: Which of the following best describes her optimal strategy, in this situation, assuming she wants to maximize her profits? Circle your choice.

i. She should stick with her initial door – that's strictly better than switching.
ii. It doesn't matter whether she switches or sticks.
iii. She should switch doors – that's strictly better than sticking.

(c) Now for a different scenario: Suppose that the contestant pays, and then Monty opens a door that contains $0. Given this, what is the expected value of her prize if she switches?

Given this, what is the expected value of her prize if she sticks with her original door?

Multiplie-choice: Which of the following best describes her optimal strategy, in this situation, assuming she wants to maximize her profits? Circle your choice.

i. She should stick with her initial door – that's strictly better than switching.
ii. It doesn't matter whether she switches or sticks.
iii. She should switch doors – that's strictly better than sticking.

(d) Now suppose a second contestant, Bob, decides in advance that he will always pay and always switch to the unopened door (no matter what he sees behind the door that Monty opens). What is the overall expected value of his prize, with this strategy?

(e) What is the most money Monty can charge for opening one of the two unchosen doors and still make it on average profitable for the contestant to pay Monty?