Last update: September 11, 2012

Informed search algorithms

CMSC 421: Chapter 3, Sections 5–6

Motivation

◇ In the previous sections we talked about trial-and-error search

◇ In the worst case, most searches take exponential time (unless P=NP)

◇ Can sometimes do much better on the average, using *heuristics*

◇ *Heuristic:*

- Rule of thumb, simplification, or educated guess
- Reduces the search for solutions in domains that are difficult and poorly understood

◇ Depending on what heuristic you use, you won't necessarily find an optimal solution, or even a solution at all

◇ We'll look at some conditions under which we're guaranteed to find optimal solutions

Heuristic tree search

```
function CAUTIOUS-TREE-SEARCH(problem)      # like TREE-SEARCH in the book
    frontier ← list that contains a node for problem's initial state
    loop
        if frontier is empty then return Failure
        choose and remove a node x from frontier
        if STATE[x] is a goal then return the corresponding solution
        else expand x and add the new nodes to frontier
```

- ◊ Heuristic choice in search algorithms: **what node to expand next**
  - Use an *evaluation function* $f(x)$ for each node x
    - ◊ estimate of how much it will cost to get to a goal
- ◊ Always choose a frontier node with the lowest f value

Heuristic graph search

function GRAPH-SEARCH(*problem*)

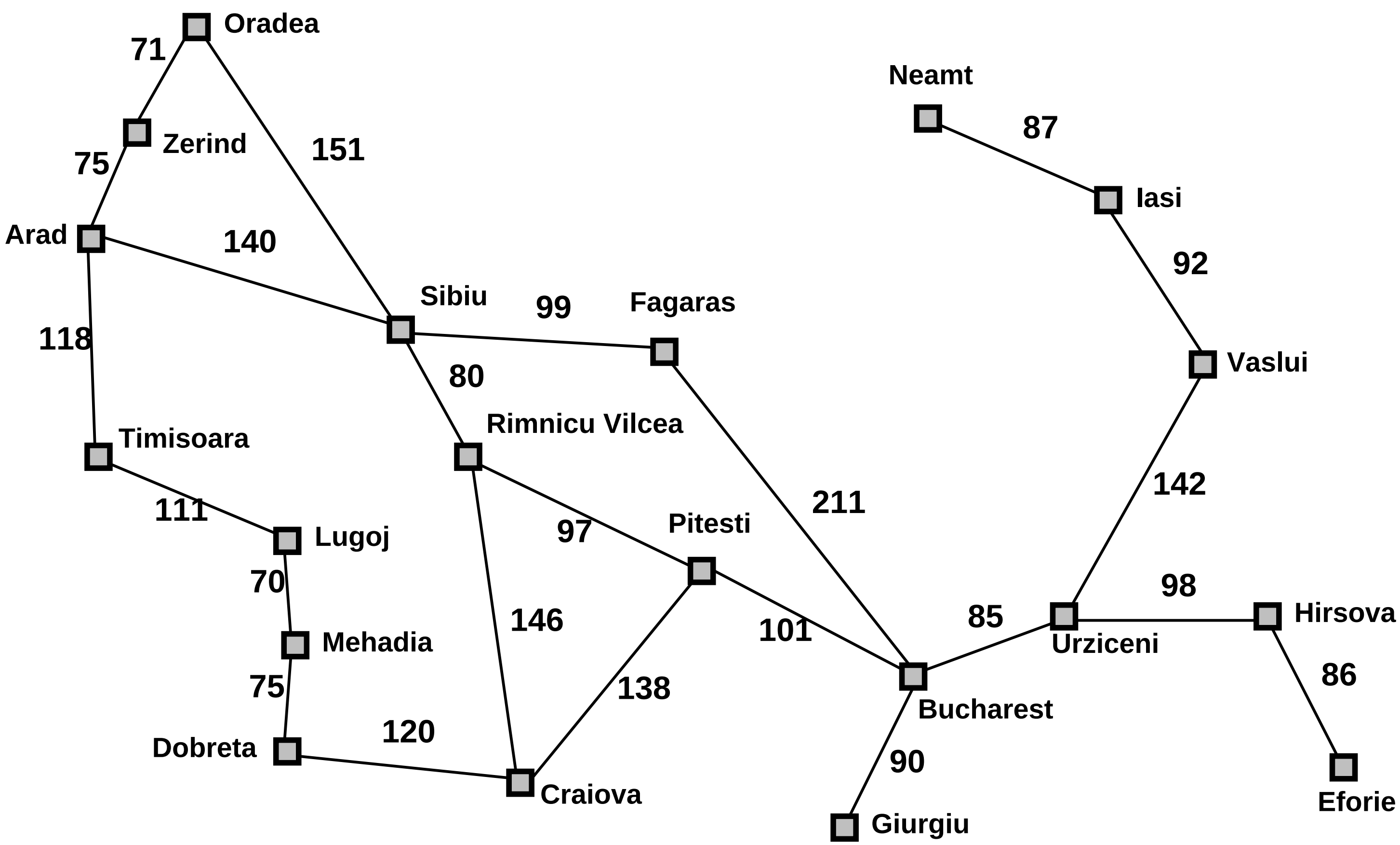
    frontier ← list that contains a node for *problem*'s initial state
    explored ← empty set
    loop
        if *frontier* is empty **then return** Failure
        choose and remove a node x from *frontier*
        if STATE[x] is a goal **then return** the corresponding solution
        if STATE[x] is not in *explored* **then**
            add STATE[x] to *explored*
            expand x and add the new nodes to *frontier*

◇ Again, always choose a frontier node with the lowest f value

Romania with step costs in km

◇ Recall that we want to get to Bucharest

◇ One possible heuristic: straight-line distance to Bucharest



Straight−line distance to Bucharest	
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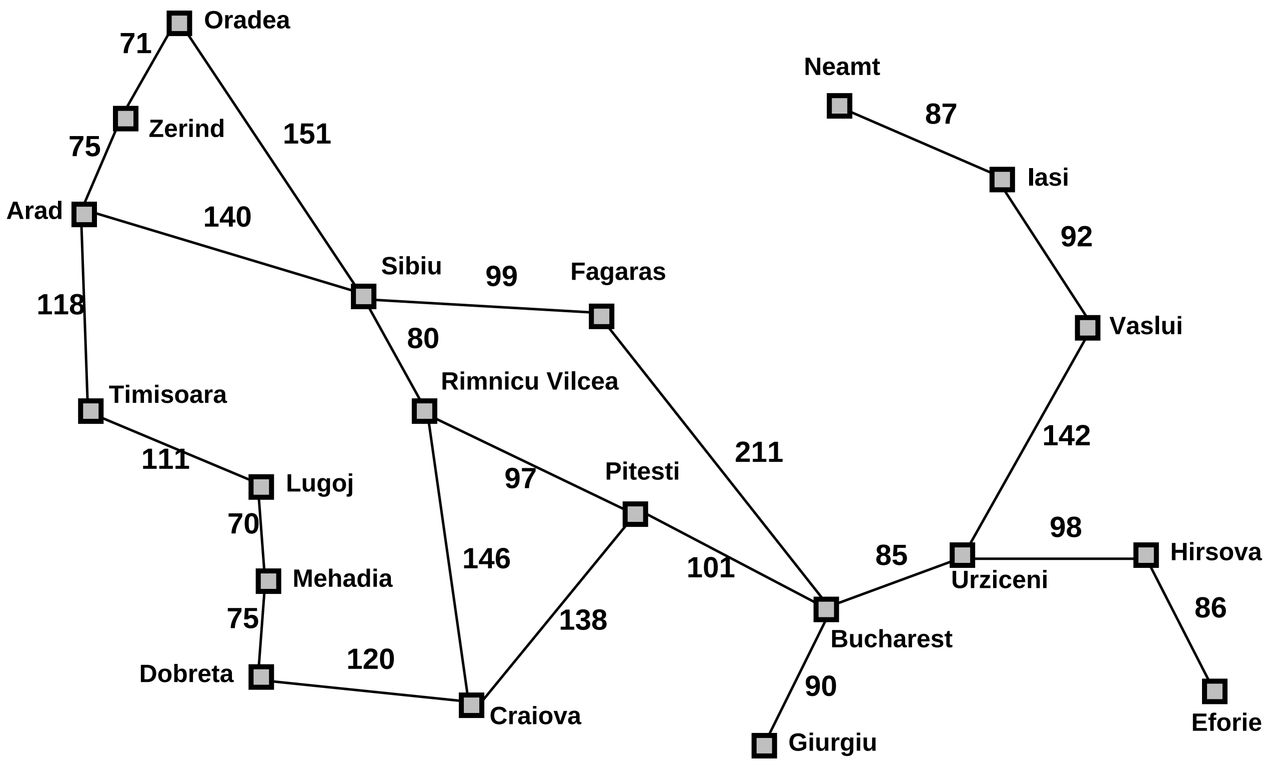
Greedy search

◊ *Heuristic function* $h(x)$ = estimate of cost from x to the closest goal

- E.g., $h_{\mathrm{SLD}}(x)$ = straight-line distance from x to Bucharest

◊ Greedy search uses $f(x) = h(x)$,

- keeps *frontier* ordered in increasing value of h
- always expands whatever node **appears** to be closest to a goal



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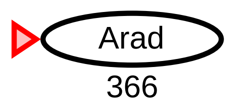

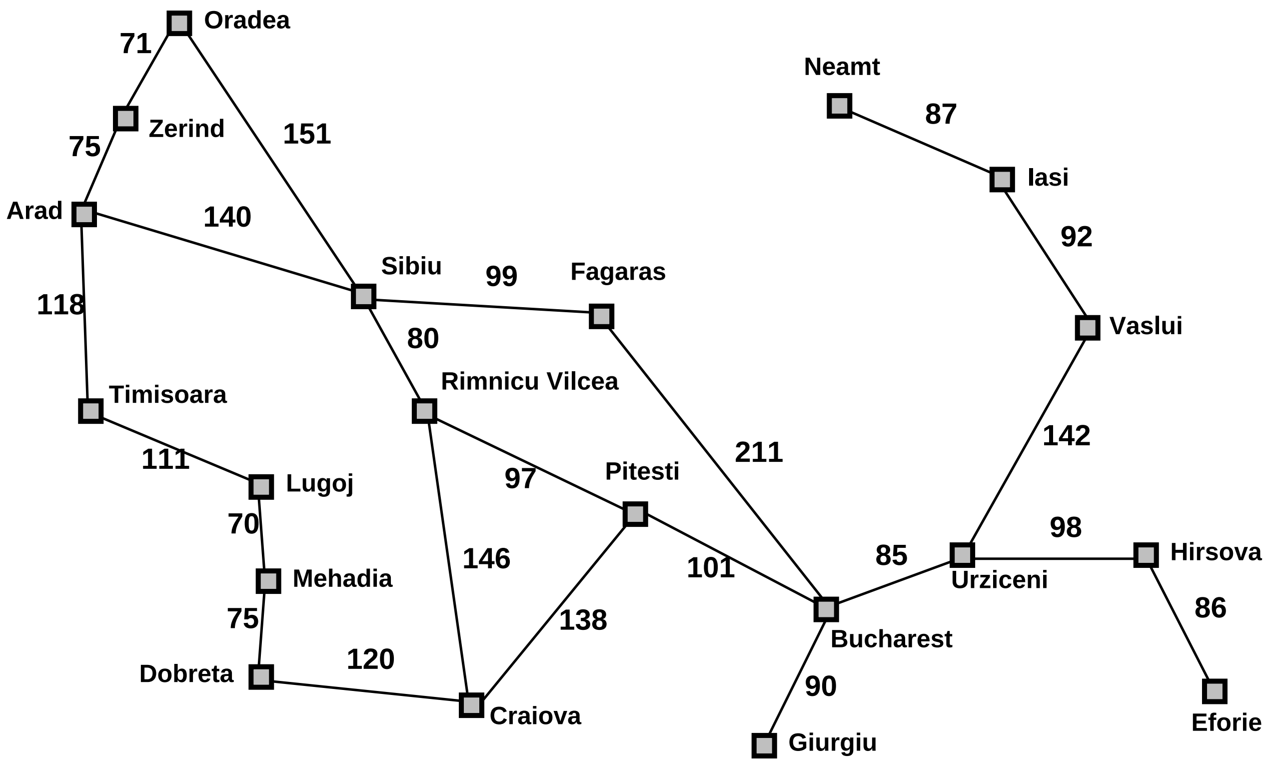

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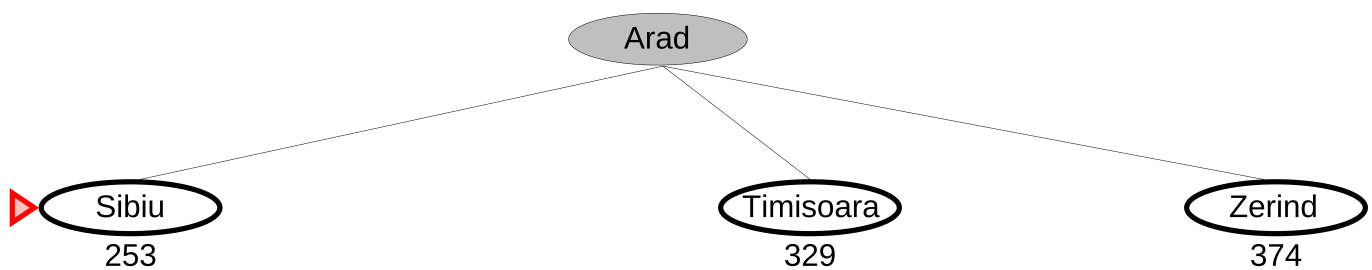

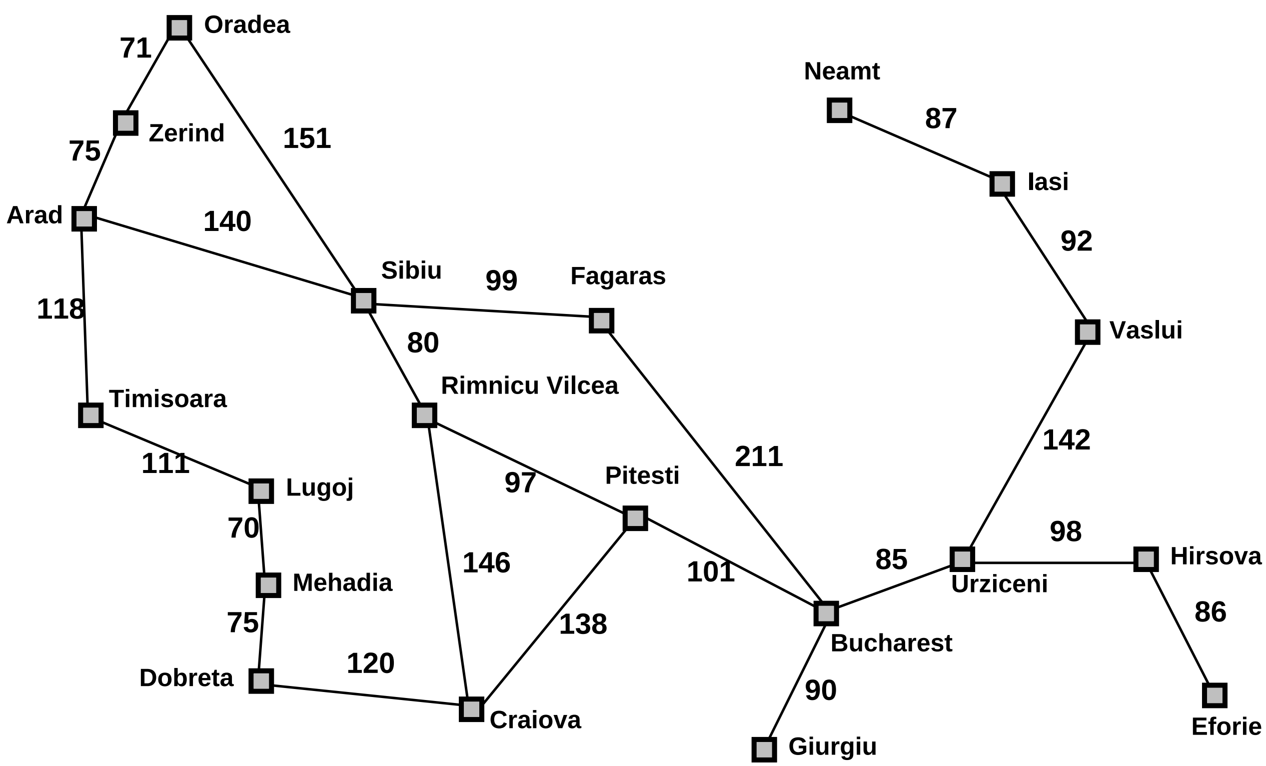

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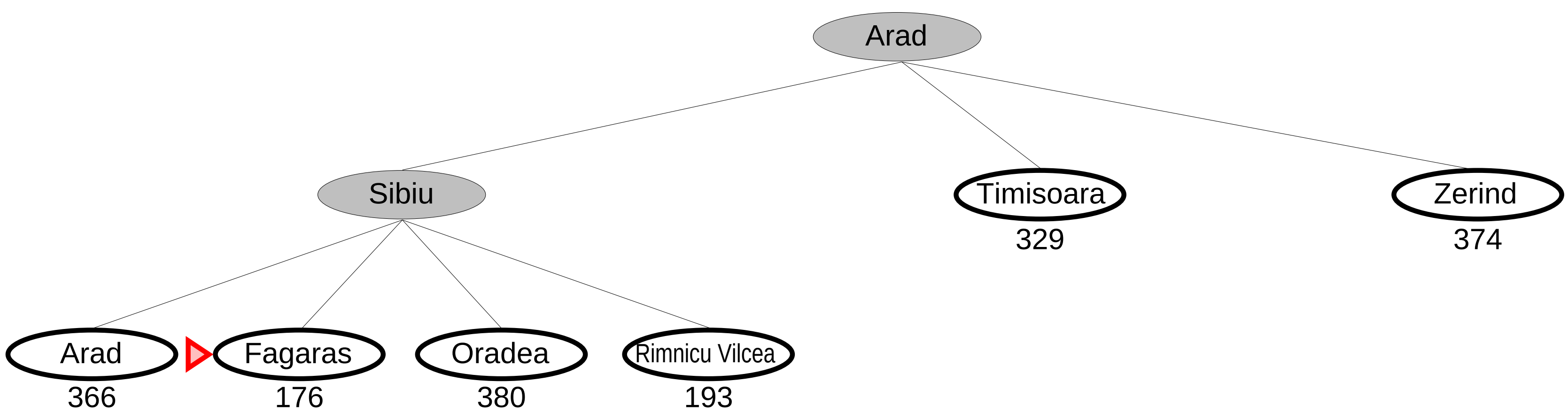

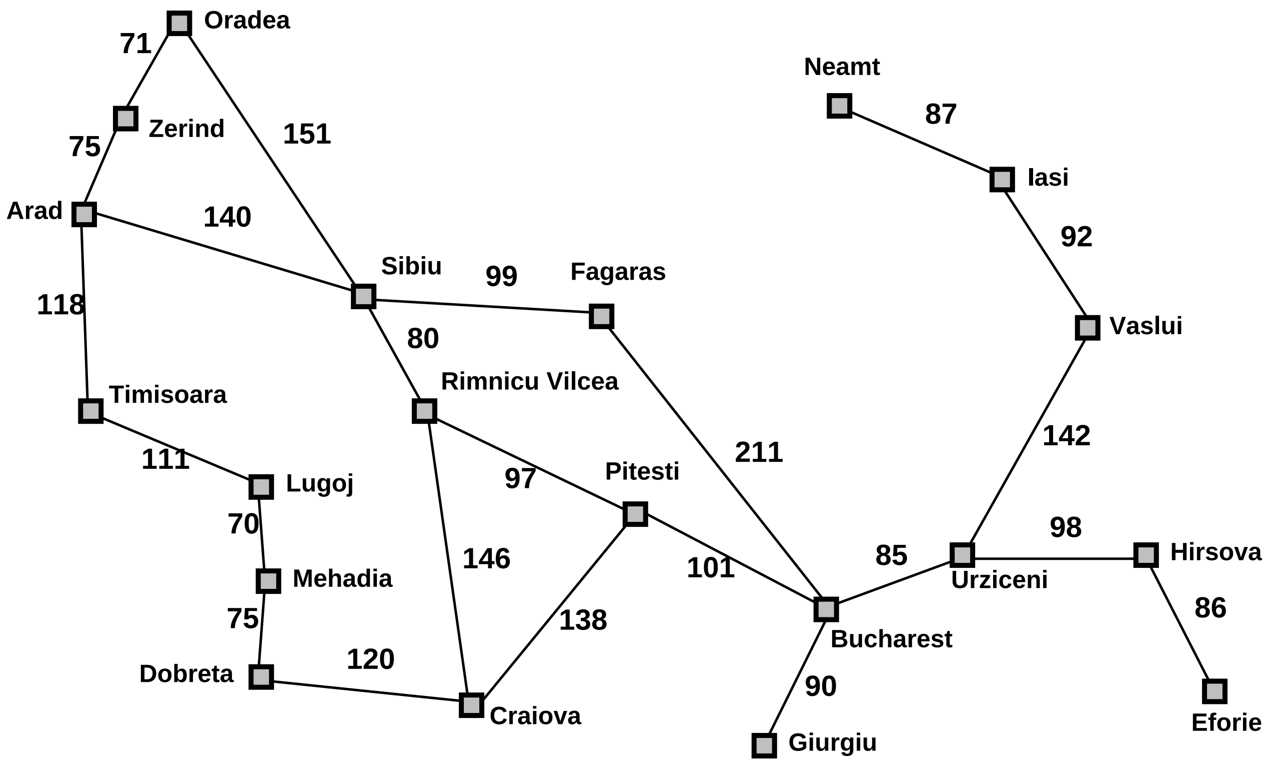

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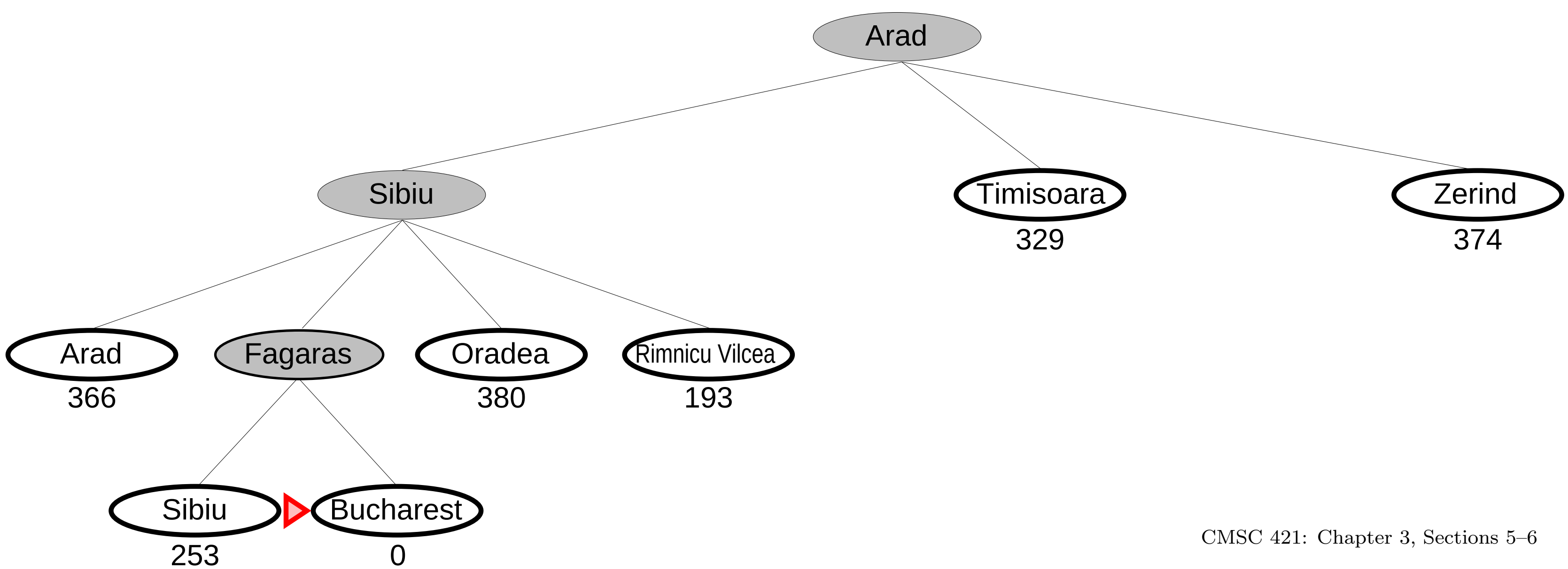

Properties of greedy search

◇ *Complete?*

Properties of greedy search

◇ *Complete?* No. Can get stuck in loops:
- Iasi → Neamt → Iasi → Neamt →
- Complete if search space is finite and we do loop-checking

◇ *Time?*

Properties of greedy search

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◊ *Time?* $O(b^m)$, but a good heuristic can give a **huge** improvement

◊ *Space?*

Properties of greedy search

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◊ *Space?* $O(b^m)$—keeps all nodes in memory

◊ *Optimal?*

Properties of greedy search

◇ *Complete?* No. Can get stuck in loops:
- Iasi → Neamt → Iasi → Neamt →
- Complete if search space is finite and we do loop-checking

◇ *Time?* $O(b^m)$, but a good heuristic can give a **huge** improvement

◇ *Space?* $O(b^m)$—keeps all nodes in memory

◇ *Optimal?* No

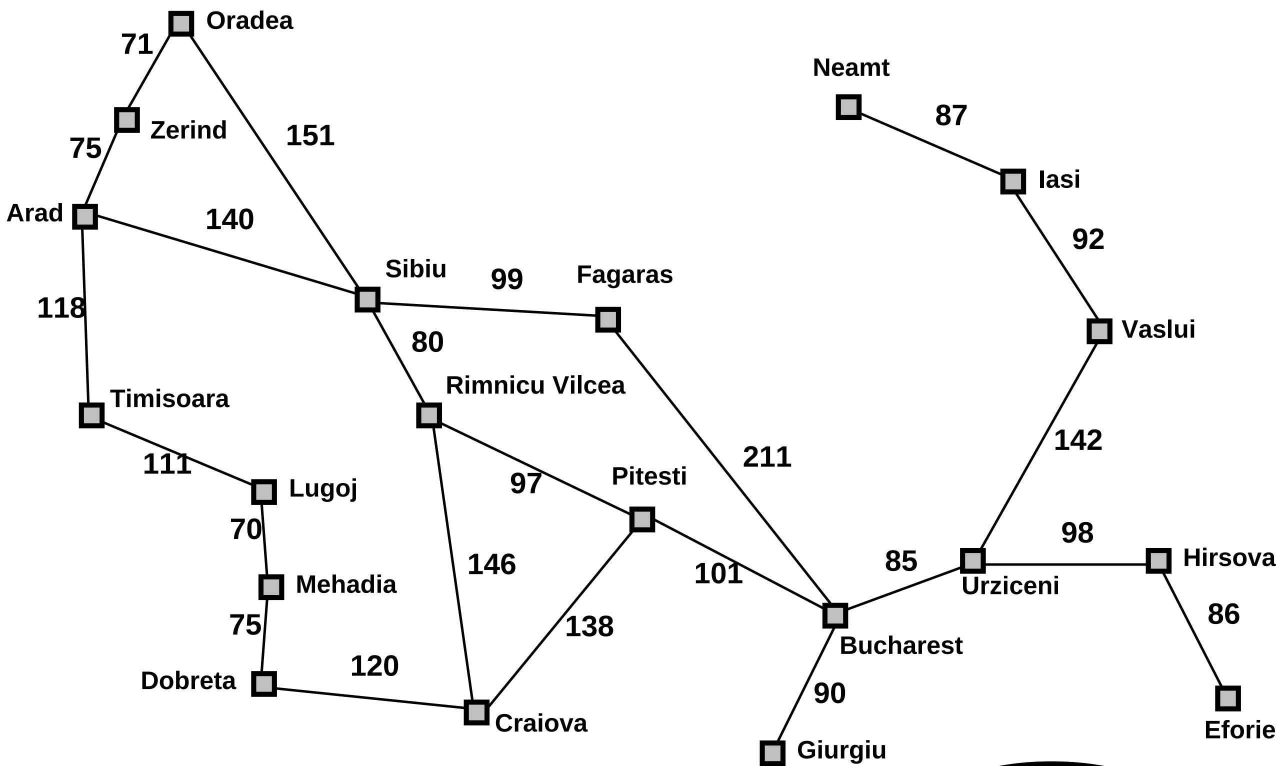
◇ A **greedy search** isn't the same as an ordinary **greedy algorithm**
- A greedy algorithm doesn't remember all of *frontier*
  - Just the current path, never goes back to look at others
  - Runs in time $O(l)$ if it finds a solution of length l
- Loop-checking cannot make it complete
  - Choose a path that doesn't lead to a solution $\Longrightarrow$ will fail
  - Normally used on problems where all paths lead to solutions

A^* tree search

◇ Idea: avoid expanding paths that are already expensive

◇ Evaluation function $f(x) = g(x) + h(x)$

- $g(x)$ = cost to reach node x
- $h(x)$ = estimated cost from x to goal
- $f(x)$ = estimated total cost of path through x to goal



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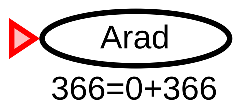
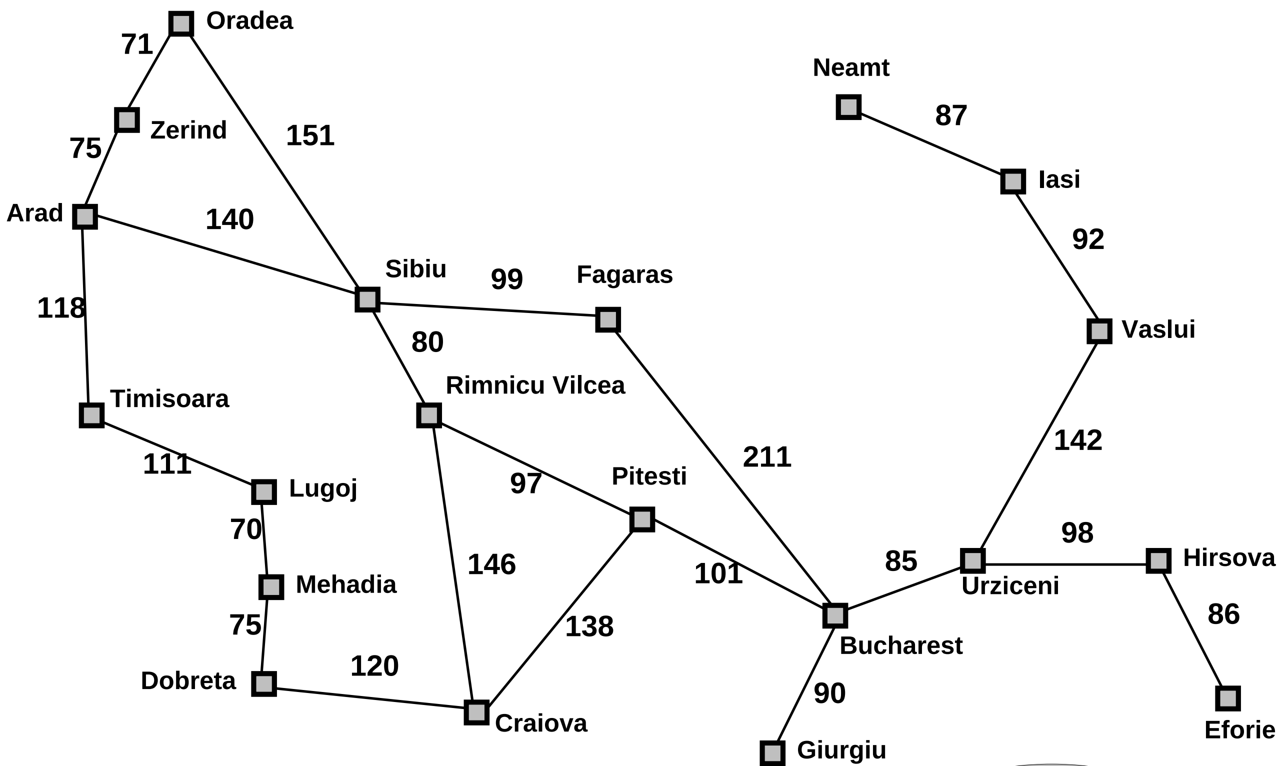
Search tree:





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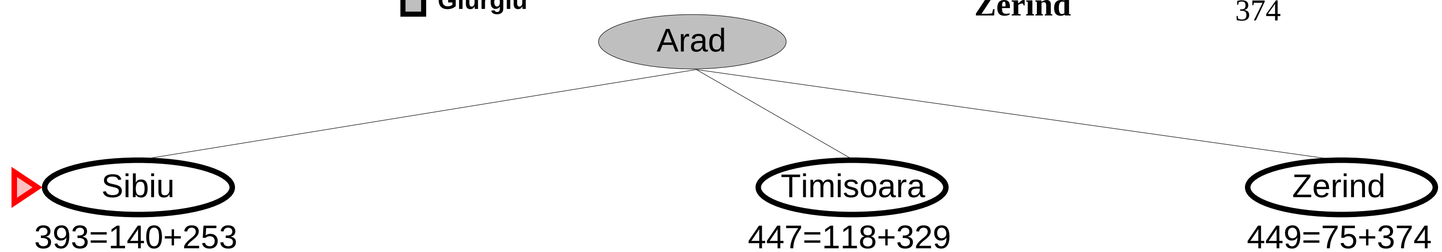

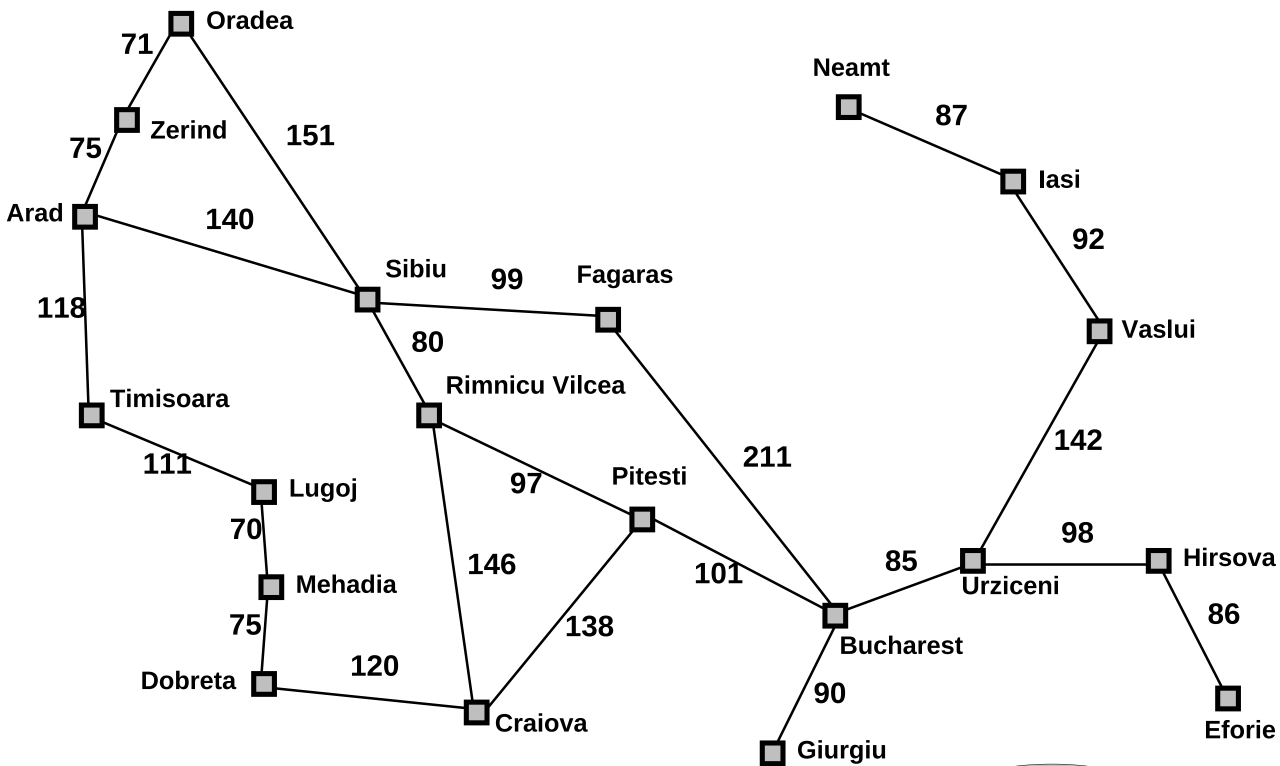

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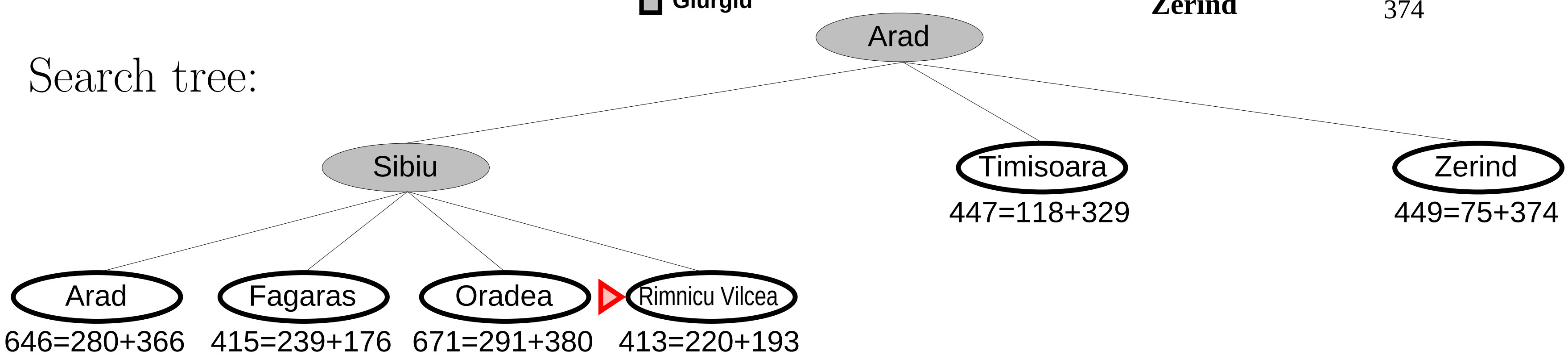

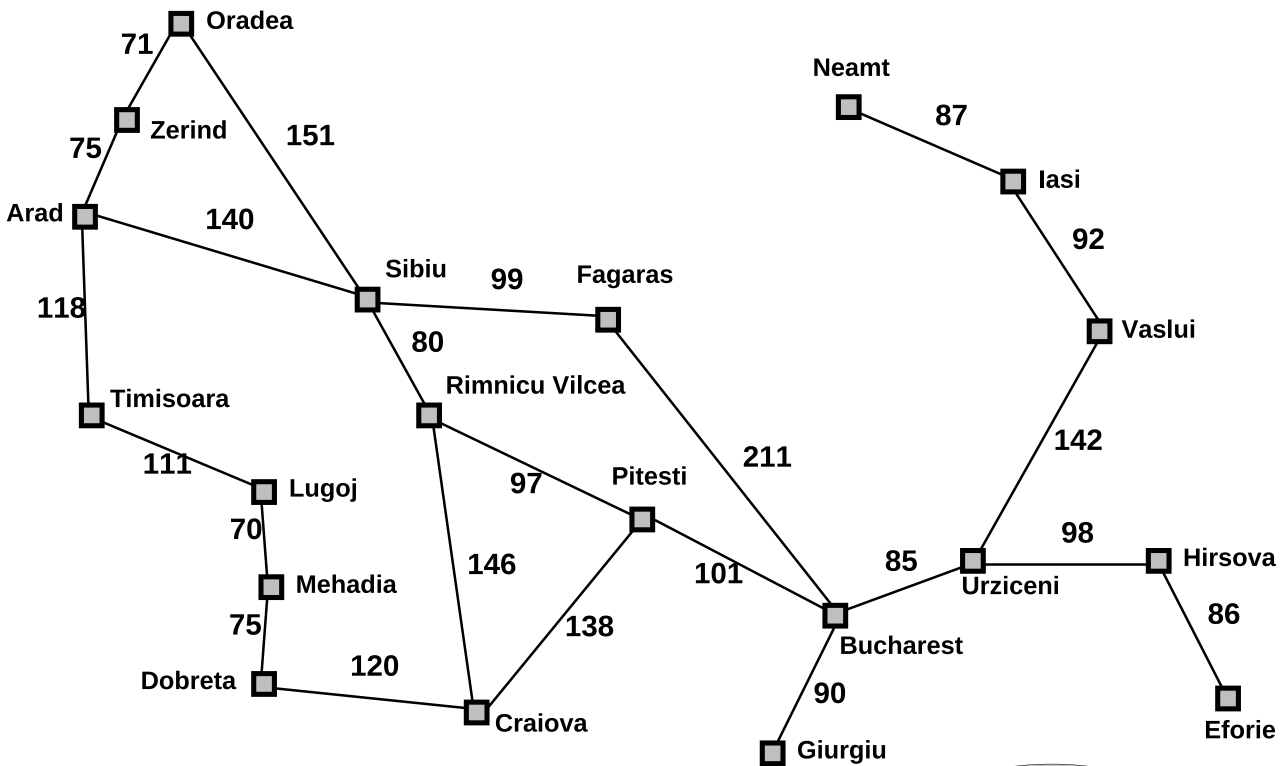

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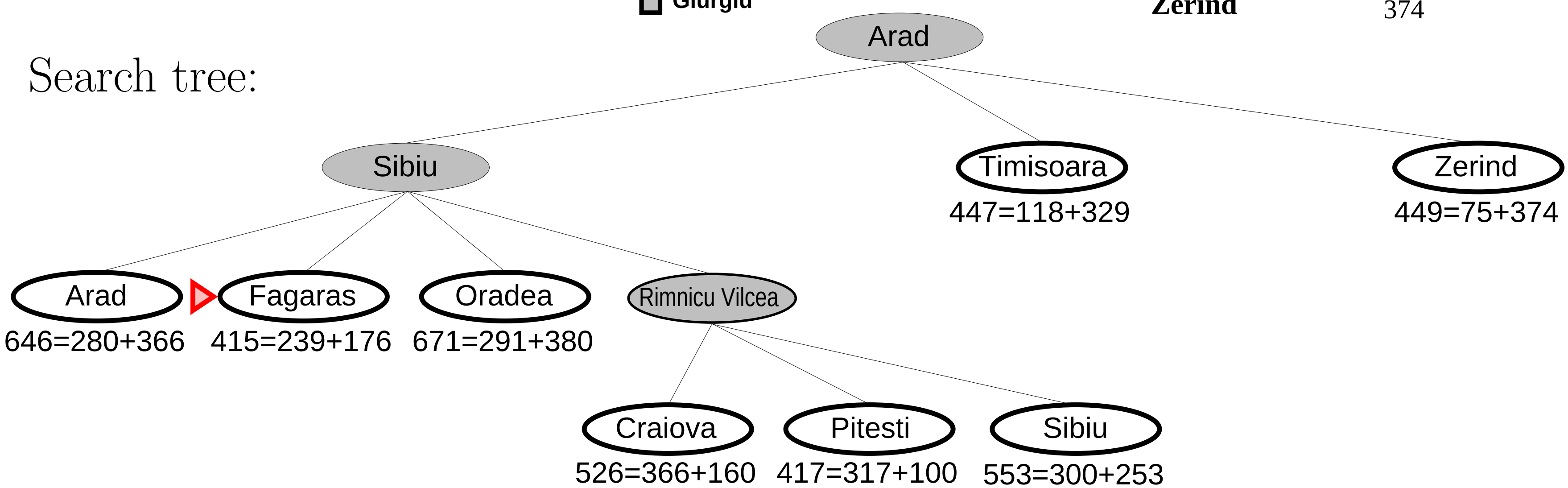

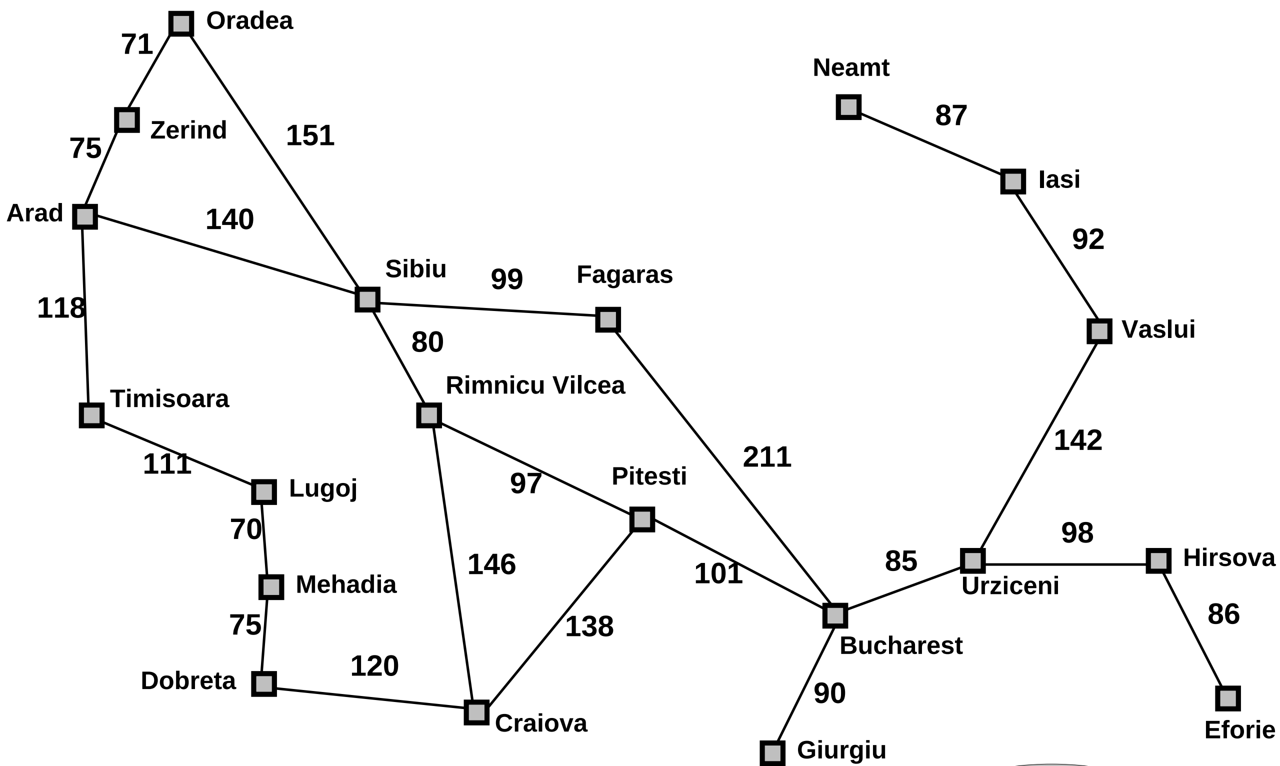

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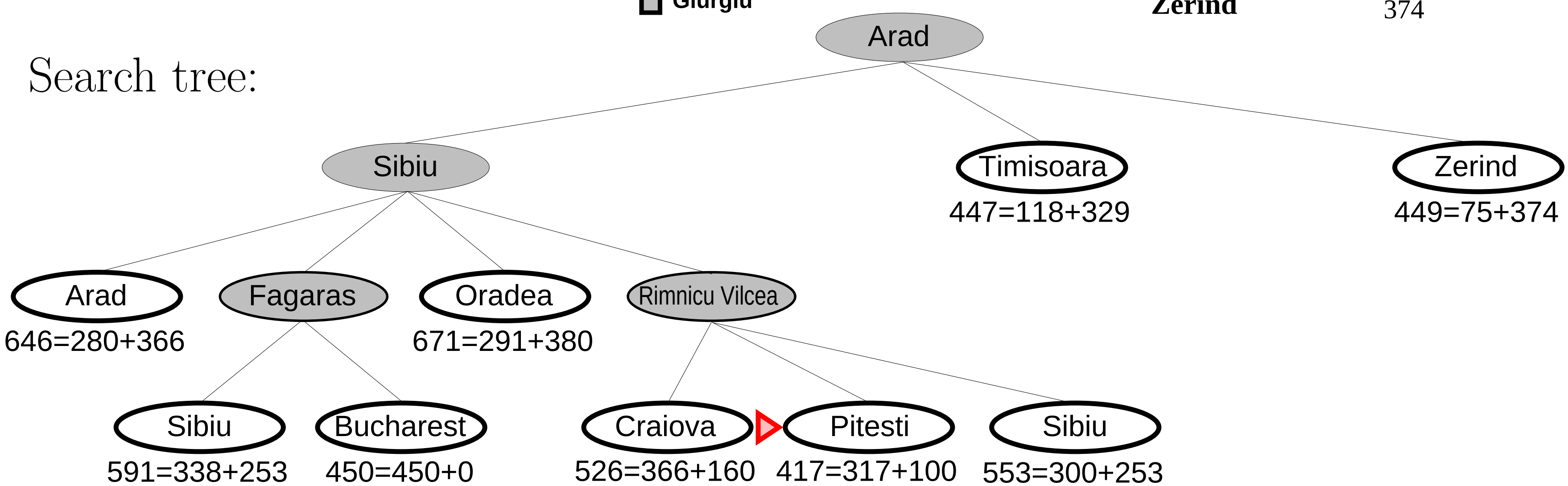

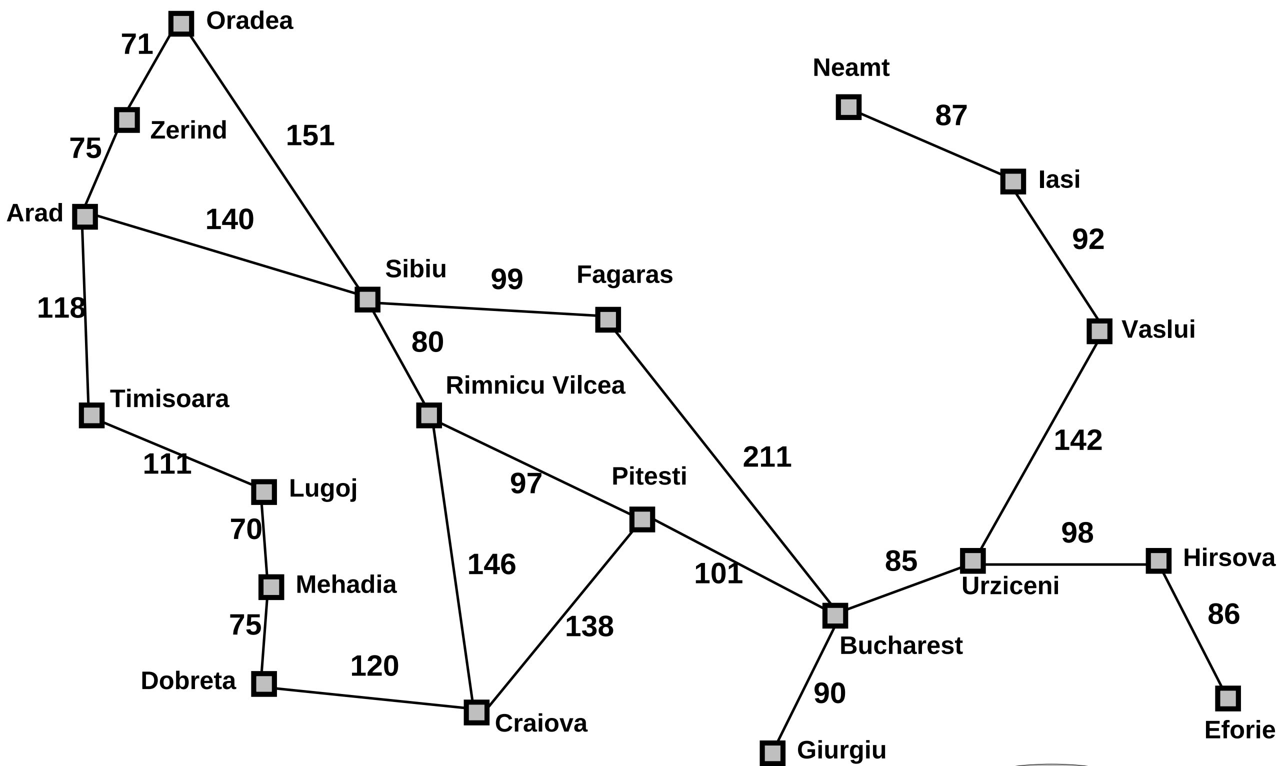

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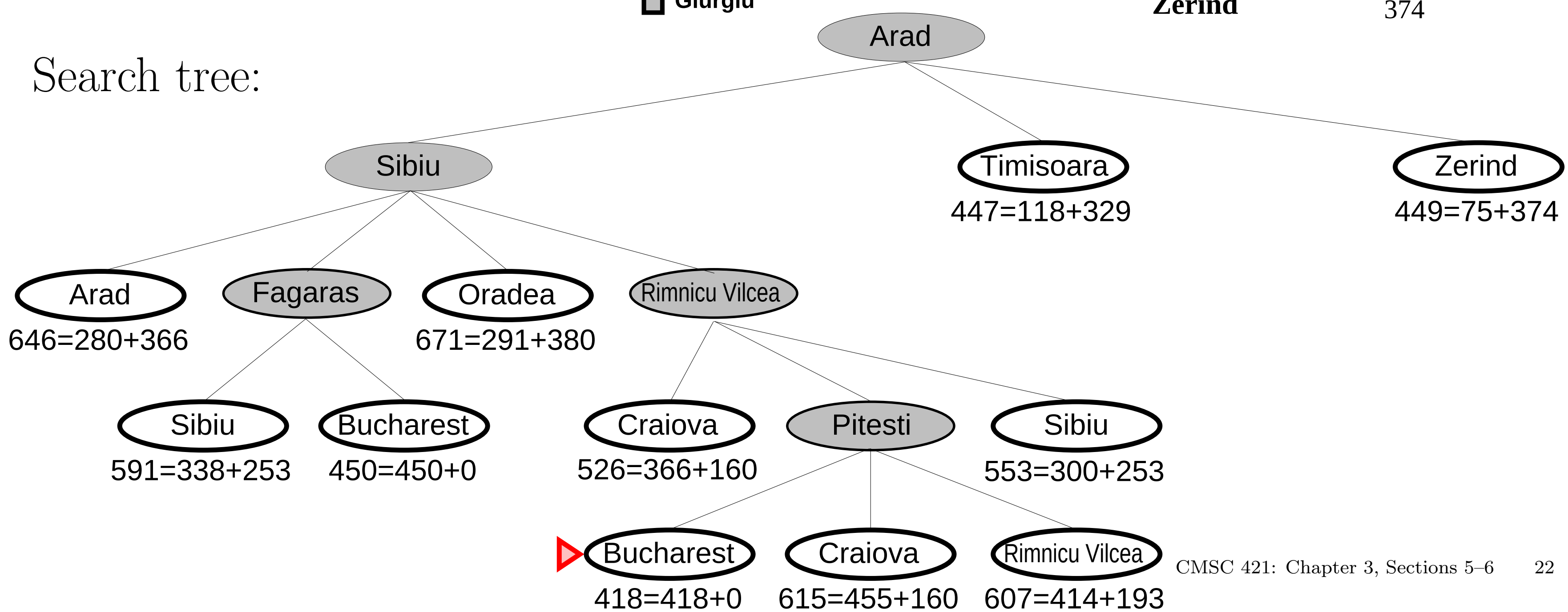


Admissibility

◇ Recall that

- $g(x)$ = cost to reach node x
- $h(x)$ = estimated cost from x to goal
- $f(x)$ = estimated total cost of path through x to goal

◇ h is *admissible* if it never overestimates the cost of getting to a goal

- $0 \leq h(x) \leq h^*(x)$ for every x,
  - ◇ where $h^*(x)$ = cost of best path from x to goal

◇ E.g., $h_{\mathrm{SLD}}(x)$ = straight-line distance from x to Bucharest

- This never overestimates the actual road distance to Bucharest

◇ If h is admissible then for every goal node x,

- $h(x) = 0$
- $f(x) = g(x)$

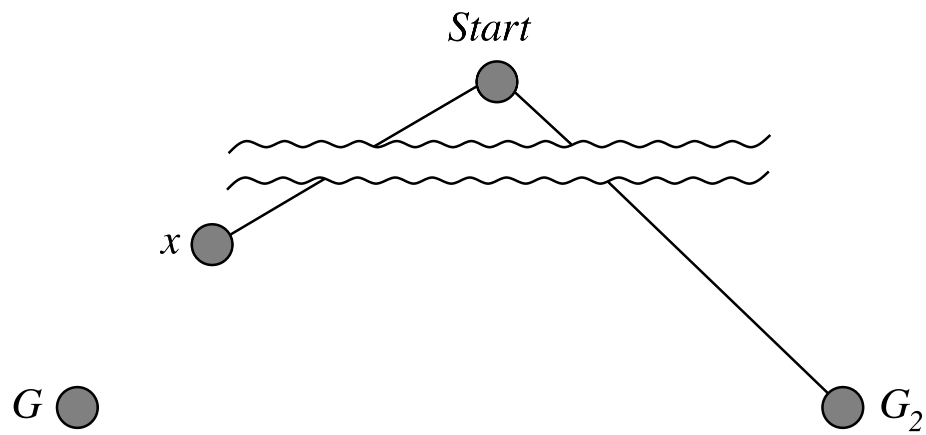
Optimality

Theorem: If h is admissible, then A* tree search will never return a non-optimal solution

◇ Let x be the first unexpanded node on an optimal path to an optimal goal node G

- Then x is in *frontier*

◇ Suppose *frontier* also contains a suboptimal goal node, G_2



$$
\begin{aligned}
f(G_2) &= g(G_2) + h(G_2) && \text{by definition} \\
&= g(G_2) && \text{since } G_2 \text{ is a goal} \\
&> g(G) && \text{since } G \text{ is optimal and } G_2 \text{ isn't} \\
&\geq g(x) + h(x) && \text{since } h \text{ is admissible} \\
&= f(x) && \text{by definition}
\end{aligned}
$$

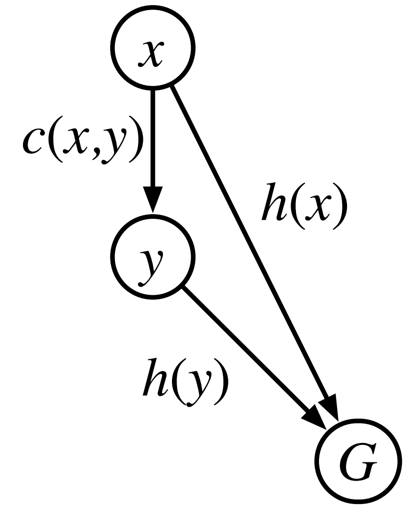
◇ A* won't select G_2, because *frontier* contains a node that looks better

Other properties

- *Completeness requirement* for A* tree search:
  - No infinite path has a finite cost
- **Theorem**: On any solvable problem that satisfies the completeness requirement, A* tree search will return a solution
- **Corollary**: If the admissibility requirement also is satisfied, then A* tree search will return an optimal solution

A* with Graph-Search

- A* can be used with Graph-Search, but there's a new requirement
- To guarantee optimality, the heuristic must be *consistent*
  - Analogous to the triangle inequality in geometry
  - For every node x and child y, $h(x) \leq c(x,y) + h(y)$



- If h is consistent, then

$$\begin{aligned} f(y) &= g(y) + h(y) \\ &= g(x) + c(x,y) + h(y) \\ &\geq g(x) + h(x) \\ &= f(x) \end{aligned}$$

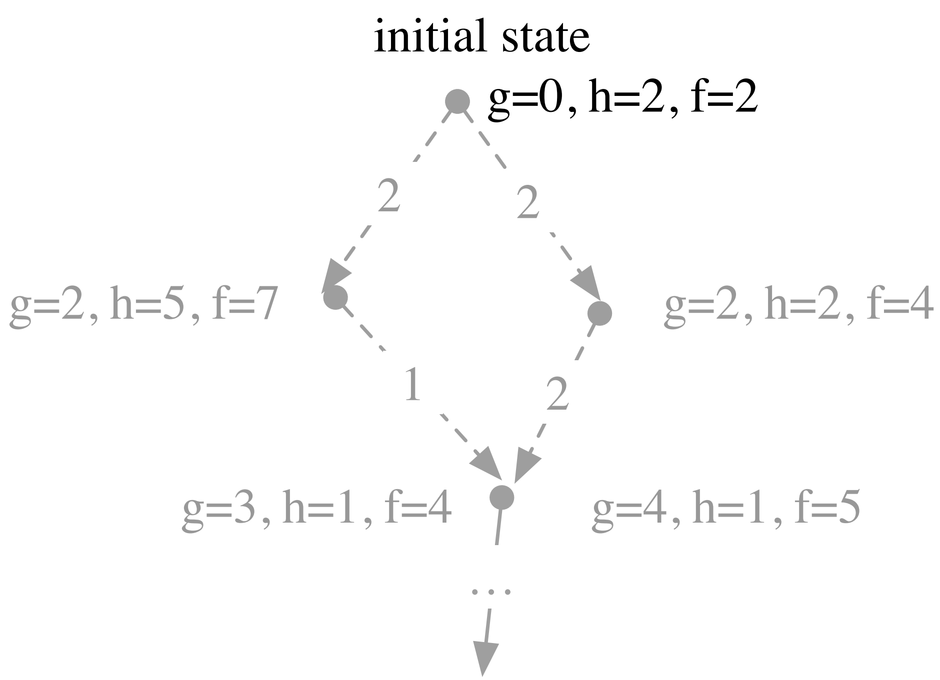
i.e., $f(x)$ is nondecreasing along any path.

- If h is consistent, then A* expands nodes in order of increasing f value

Behavior of A* with inconsistent heuristic

◇ If h is inconsistent, then

- As we go along a path, f may sometimes decrease
- A* may find lower-cost paths to states it has already expanded



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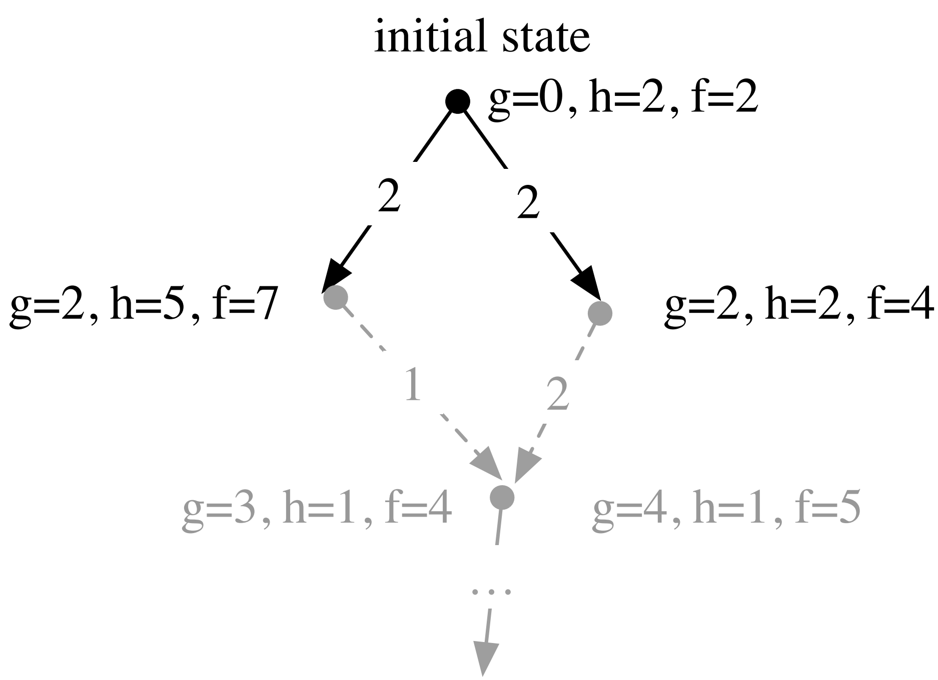

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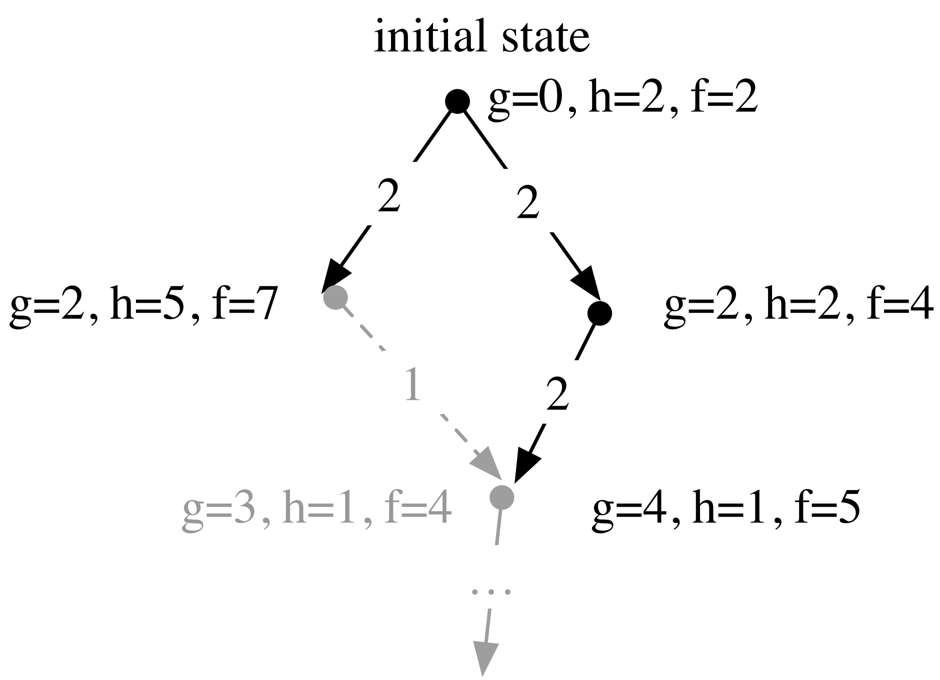

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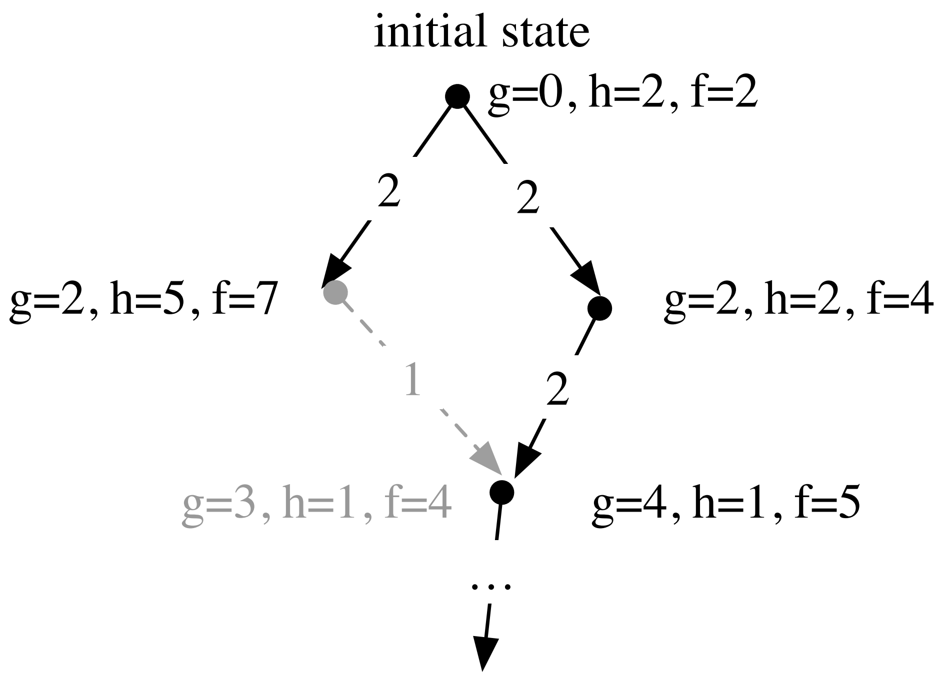

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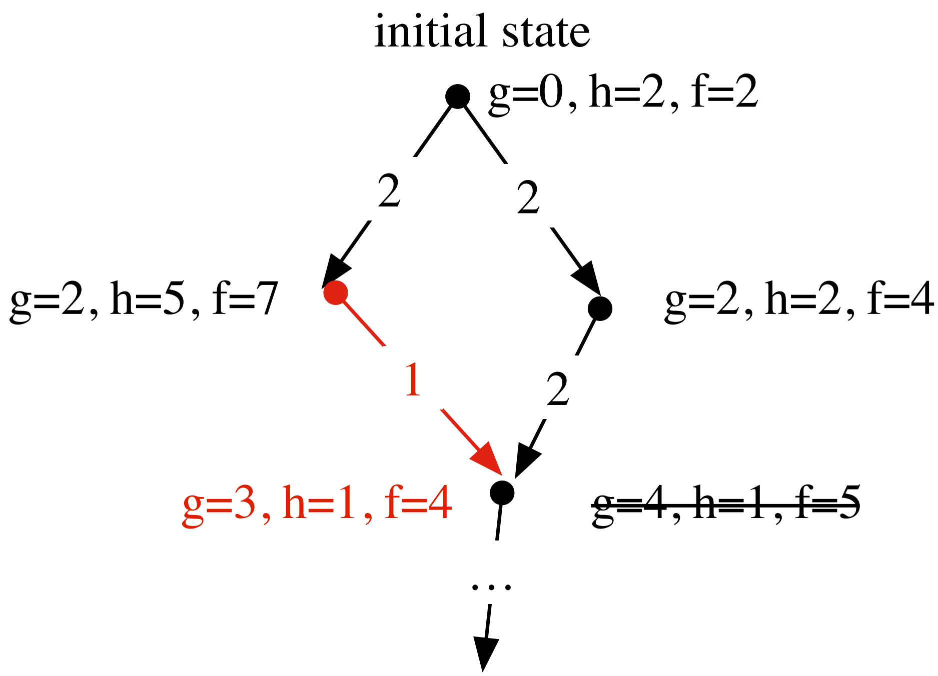


◇ Need to re-expand, but GRAPH-SEARCH won't do so

A* for graphs

◇ Find optimal solutions even if the heuristic is inconsistent

- Make *explored* a list of nodes, so we know their f values
- Re-expand states if better paths are found

function A*-GRAPH-SEARCH(*problem*)

explored ← an empty set
frontier ← list that contains a node for *problem*'s initial state
loop
  if *frontier* is empty **then return** Failure
  $x \leftarrow \arg\min\{f(y) \mid y \in \mathit{frontier}\}$
  if STATE[x] is a goal state **then return** x
  if there is no node y in *explored* such that
    STATE[y] = STATE[x] and $f(y) \leq f(x)$
  then
    add x to *explored*
    expand x and add the new nodes to *frontier*

Properties of A*

◊ *Complete?*

Properties of A*

- ◊ *Complete?* Yes, unless there are infinitely many nodes with $f \leq f(G)$
- ◊ *Time?*

Properties of A*

◇ <u>*Complete?*</u> Yes, unless there are infinitely many nodes with $f \leq f(G)$

◇ <u>*Time?*</u> O(entire state space) in worst case, $O(d)$ in best case

◇ <u>*Space?*</u>

Properties of A*

- <u>*Complete?*</u> Yes, unless there are infinitely many nodes with $f \leq f(G)$
- <u>*Time?*</u> O(entire state space) in worst case, $O(d)$ in best case
- <u>*Space?*</u> Keeps all nodes in memory
- <u>*Finds optimal solutions?*</u>

Properties of A*

- *Complete?* Yes, unless there are infinitely many nodes with $f \le f(G)$
- *Time?* O(entire state space) in worst case, $O(d)$ in best case
- *Space?* Keeps all nodes in memory
- *Finds optimal solutions?* Yes on trees
  - Also on graphs, if the heuristic is consistent
    or if we use the rewritten version of A*

- Additional properties:
  - A* expands all nodes in *frontier* that have $f(x) < C^*$
  - A* expands some nodes with $f(x) = C^*$
  - If f is *consistent*, A* expands no nodes with $f(x) > C^*$

How to create admissible heuristics

◇ Suppose P is a problem we're trying to solve

- Let $h^*(x)$ = minimum cost of any solution path from x

◇ Let Q be a **relaxation** of P

- Remove some constraints on what constitutes a solution

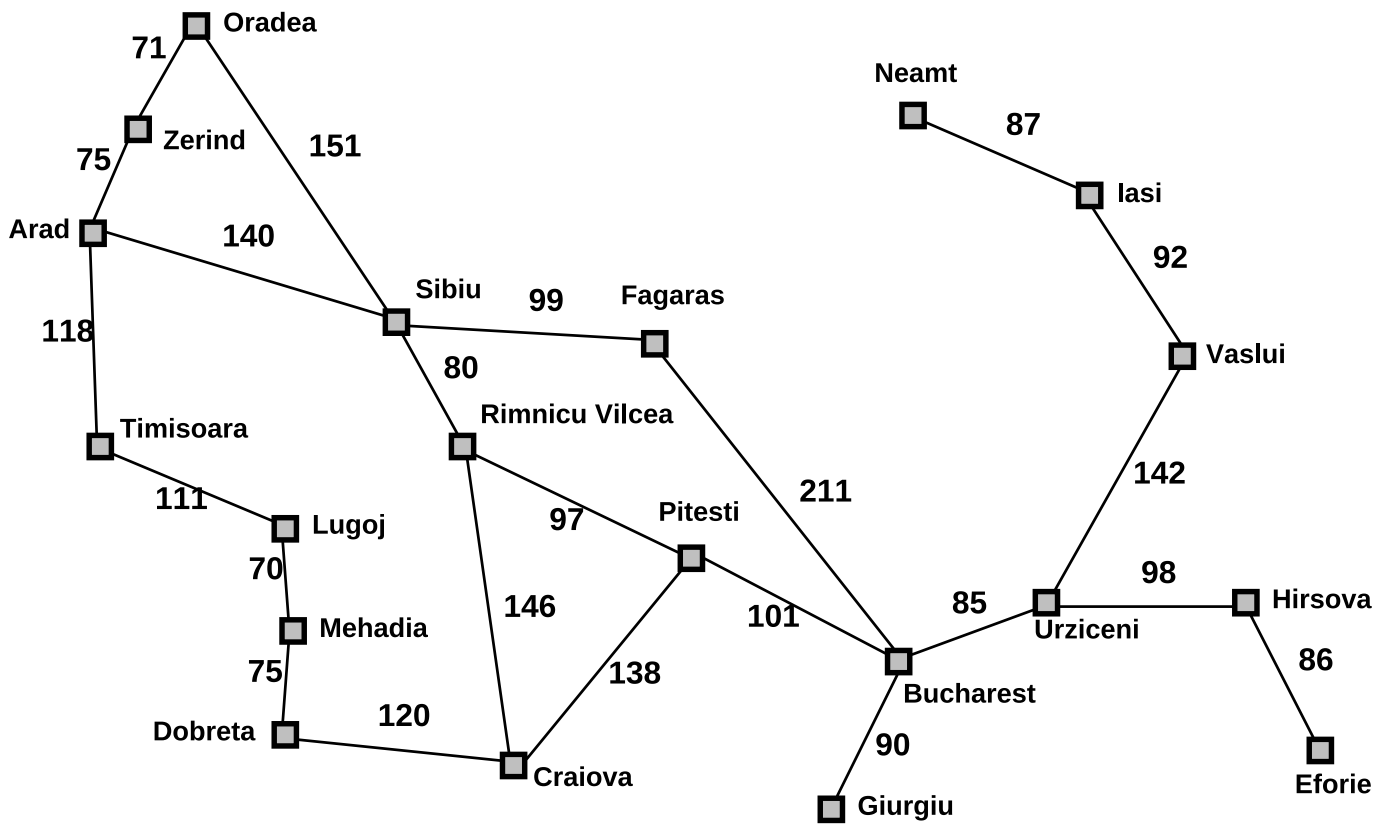
◇ Every solution path from x in P is also a solution path in Q

- Q may have some additional solution paths
- Some of them may cost less than $h^*(x)$

◇ Suppose that in Q, we can find optimal solutions quickly

- At x, find an optimal solution in Q
- Let $h(x)$ = cost of the solution we found
- Then $h(x) \leq h^*(x)$, i.e., h is an admissible heuristic for P

Example: Romania with step costs in km



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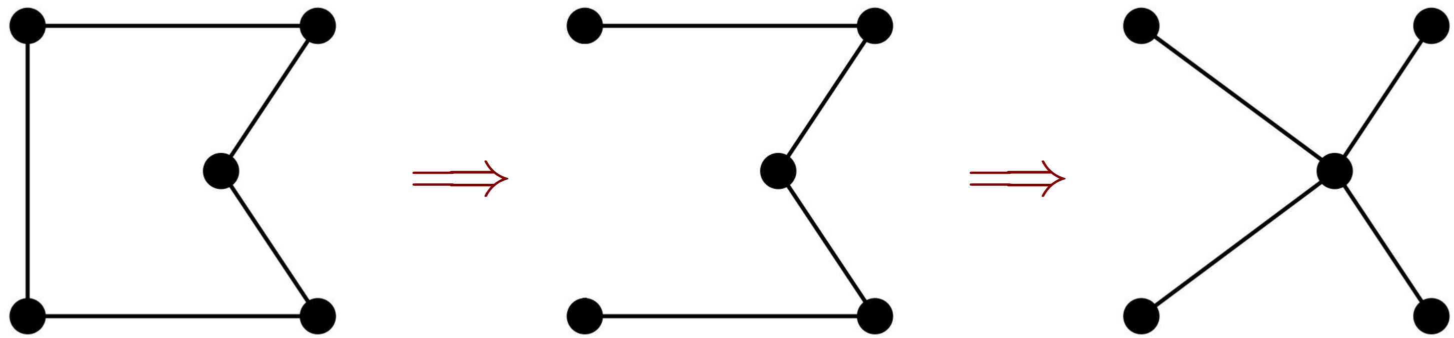
◇ Relax the problem to allow straight-line paths

- $h_{\text{SLD}}(x)$ cost of a straight line from city x to Bucharest

Example: TSP

- Well-known example: *traveling salesperson problem* (TSP)
  - Given a *complete* graph (edges between all pairs of nodes)
  - Find a least-cost *tour* (simple cycle that visits each city exactly once)



- Relax the problem twice:
  - Let {solutions} include acyclic paths that visit all cities
  - Let {solutions} include trees
- *Minimum spanning tree* can be computed in $O(n^2)$

  $\Rightarrow$ lower bound on the least-cost path that visits all cities

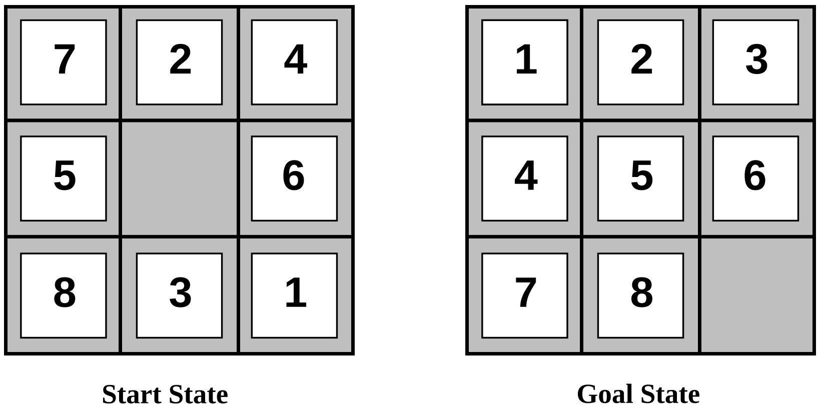
  $\Rightarrow$ lower bound on the least-cost tour

Example: the 8-puzzle

◇ Relaxation 1: allow a tile to move to any other square

- regardless of whether the square is adjacent
- regardless of whether there's another tile there already

◇ This gives us $h_1(x)$ = number of misplaced tiles



$h_1(S)$ =?

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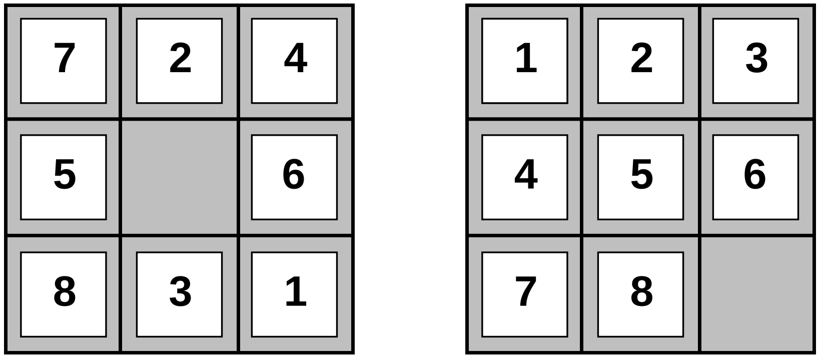
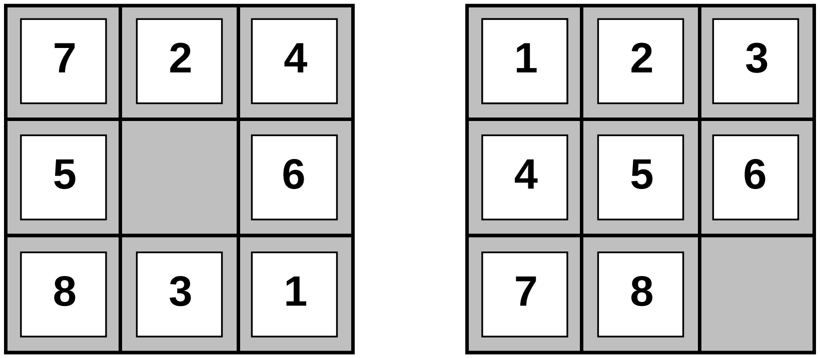


Start State Goal State

$h_1(S)$ =? 6

Example: the 8-puzzle

- ◇ Relaxation 2: allow a tile to move to any *adjacent* square
  - regardless of whether there's another tile there already
- ◇ This gives us $h_2(x)$ = total *Manhattan* distance



Start State **Goal State**

$h_2(S)$ =?

Example: the 8-puzzle

- ♢ Relaxation 2: allow a tile to move to any adjacent square
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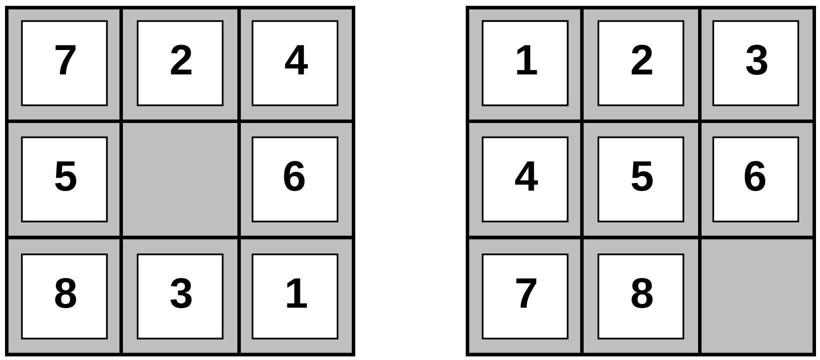

Start State **Goal State**

$h_2(S)$ =? $4+0+3+3+1+0+2+1 = 14$

Dominance

◇ $h_1(x)$ = number of misplaced tiles

◇ $h_2(x)$ = total *Manhattan* distance

◇ h_2 *dominates* h_1:

- $h_1(x) \leq h_2(x) \leq h^*(x)$ for all x,

◇ h_2's estimate of h^* is at least as good as h_1's, and often better

- Hence h_2 is better for search. Typical search costs:

$d = 14$:	IDS	3,473,941 nodes	
	A*(h_1)	539 nodes	
	A*(h_2)	113 nodes	
$d = 24$:	IDS	54,000,000,000 nodes	(approx.)
	A*(h_1)	39,135 nodes	
	A*(h_2)	1,641 nodes	

How to get dominance?

◇ If h_a are h_b admissible heuristic functions, then

- $h(x) = \max(h_a(x), h_b(x))$ is admissible and dominates both h_a and h_b

How to *fake* dominance:

◇ $h_2(x) = ch_1(x)$, where c is a positive constant

◇ Probably h_2 won't be admissible

$\Longrightarrow$ Won't always get optimal solutions

- But you may be able to get solutions much faster

◇ The larger you make c, the closer you get to a greedy search

Iterative-Deepening A*

function IDA*(*problem*) **returns** a solution
    inputs: *problem*, a problem

    $fmax \leftarrow h(initial\ state)$
    for $i \leftarrow 0$ **to** ∞ **do**
        $result \leftarrow$ LIMITED-F-SEARCH(*problem, fmax*)
        if *result* is a solution **then return** *result*
        else $fmax \leftarrow$ result
    end

function LIMITED-F-SEARCH(*problem, fmax*) **returns** solution or number

    depth-first search, backtracking at every node n such that $f(n) > fmax$
    if the search finds a solution
    then return the solution
    else return $\min\{f(m) \mid \text{the search backtracked at node } m\}$

Properties of IDA*

◊ *Complete?*

Properties of IDA*

◊ *Complete?* Yes, unless there are infinitely many nodes with $f(x) \leq f(G)$

◊ *Time?*

Properties of IDA*

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- *Time?* Like A* **if** $f(x)$ is an integer and
  the number of nodes with $f(x) \leq k$ grows exponentially with k
- *Space?*

Properties of IDA*

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- *Space?* $O(bd)$
- *Optimal?*

Properties of IDA*

◊ *Complete?* Yes, unless there are infinitely many nodes with $f(x) \leq f(G)$

◊ *Time?* Like A* **if** $f(x)$ is an integer and
the number of nodes with $f(x) \leq k$ grows exponentially with k

◊ *Space?* $O(bd)$

◊ *Optimal?* Yes

◊ With consistent heuristic:

- IDA* cannot expand f_{i+1} until f_i is finished
- IDA* expands all nodes with $f(x) < C^*$
- IDA* expands no nodes with $f(x) \geq C^*$

Summary

◇ Heuristic functions estimate costs of shortest paths

◇ Good heuristics can dramatically reduce search cost

◇ Greedy search expands lowest h

- incomplete, solutions not always optimal

◇ A* search expands lowest $g + h$

- Completeness and optimality if some requirements are satisfied
  - ◇ admissibility, etc.
- Can get admissible heuristics by solving relaxed problems

◇ IDA* is like a combination of A* and IDS

- much lower space requirement than A*
- same big-O time *if* number of nodes grows exponentially with cost