such that for each $i \leq n$ we have at least one of

ε_i is a sentence symbol

$\varepsilon_i = \mathcal{E}_{\neg}(\varepsilon_j)$ for some $j < i$

$\varepsilon_i = \mathcal{E}_{\square}(\varepsilon_j, \varepsilon_k)$ for some $j < i, k < i$

where $\square$ is one of the binary connectives, $\wedge, \vee, \rightarrow, \leftrightarrow$. Then the well-formed formulas can be characterized as the expressions α such that some construction sequence ends with α. We can think of ε_i as the expression at stage i in the building process.

For our earlier example,

$$((\mathbf{A}_1 \wedge \mathbf{A}_{10}) \rightarrow ((\neg \mathbf{A}_3) \vee (\mathbf{A}_8 \leftrightarrow \mathbf{A}_3)))$$

we obtain a construction sequence by squashing its ancestral tree into a linear ordering.

One feature of this sort of construction is that it yields an *induction principle*. Say that a set S is *closed* under a two-place function f iff whenever $x \in S$ and $y \in S$ then $f(x, y) \in S$, and similarly for one-place functions and so forth.

Induction Principle If S is a set of wffs containing all the sentence symbols and closed under all five formula-building operations, then S is the set of *all* wffs.

First Proof. Consider an arbitrary wff α. It is built up from sentence symbols by applying some finite number of times the formula-building operations. Working our way up the corresponding ancestral tree, we find that each expression in the tree belongs to S. Eventually (that is, after a finite number of steps) at the top of the tree we find that $\alpha \in S$. ⊣

Second Proof. We will repeat the argument, but without the trees. Consider an arbitrary wff α. It is the last member of some construction sequence $\langle \varepsilon_1, \ldots, \varepsilon_n \rangle$. By ordinary strong numerical induction on the number i, we see that each $\varepsilon_i \in S$, $i \leq n$.

That is, we suppose, as our inductive hypothesis, that $\varepsilon_j \in S$ for all $j < i$. We then verify that $\varepsilon_i \in S$, by considering the various cases. So by strong induction on i, it follows that $\varepsilon_i \in S$ for each $i \leq n$. In particular, the last member α belongs to S. ⊣

This principle will receive much use in the coming pages. In the following example we use it to show that certain expressions are *not* wffs.

Example. Any expression with more left parentheses than right parentheses is not a wff.

Proof. The idea is that we start with sentence symbols (hav
zero left parentheses and zero right parentheses), and then
ply formula-building operations which add parentheses onl
matched pairs. We can rephrase this argument as follows:
set of "balanced" wffs (having equal numbers of left and r
parentheses) contains all sentence symbols and is closed u
the formula-building operations. The induction principle ther
sures us that all wffs are balanced.

A special feature of our particular formula-building operations is
they build *up* and never *down*. That is, the expression $\mathcal{E}_{\square}(\alpha, \beta)$ alv
includes as a segment the entire sequence α (and the entire sequenc
plus other symbols. In particular, it is longer than either α or β.

This special feature will simplify the problem of determining, g
a wff φ, exactly *how* it was built up. All the building blocks, so to sp
are included as segments in the sequence φ. For example, if φ doe
contain the symbol $\mathbf{A}_4$, then φ can be built up without ever using
(See Exercise 4.)

Exercises

1. Give three sentences in English together with translations int
formal language. The sentences should be chosen so as to hav
interesting structure, and the translations should each contain
more symbols.
2. Show that there are no wffs of length 2, 3, or 6, but that any
positive length is possible.
3. Let α be a wff; let c be the number of places at which binary
nective symbols ($\wedge, \vee, \rightarrow, \leftrightarrow$) occur in α; let s be the numb
places at which sentence symbols occur in α. (For example, i
$(\mathbf{A} \rightarrow (\neg \mathbf{A}))$ then $c = 1$ and $s = 2$.) Show by using the indu
principle that $s = c + 1$.
4. Assume we have a construction sequence ending in φ, where φ
not contain the symbol $\mathbf{A}_4$. Suppose we delete all the expres
in the construction sequence that contain $\mathbf{A}_4$. Show that the res
still a legal construction sequence.
5. Suppose that α is a wff not containing the negation symbol $\neg$.
(a) Show that the length of α (i.e., the number of symbols i
string) is odd.
(b) Show that more than a quarter of the symbols are sen
symbols.

Suggestion: Apply induction to show that the length is of the
$4k + 1$ and the number of sentence symbols is $k + 1$.