Truth Assignments and Truth Tables

Outline

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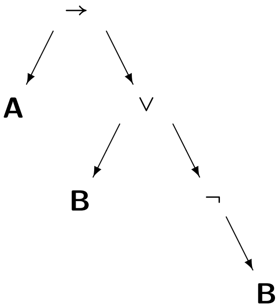
  - Infinite sets? (later...)

Outline

Example

Consider $\tau = (\mathbf{A} \rightarrow (\mathbf{B} \vee (\neg \mathbf{B})))$.



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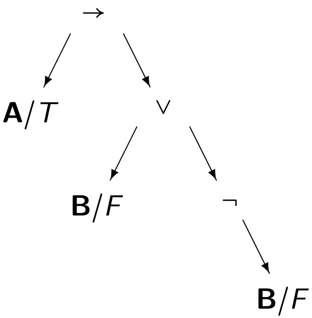

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Work truth values up tree, using common semantics of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

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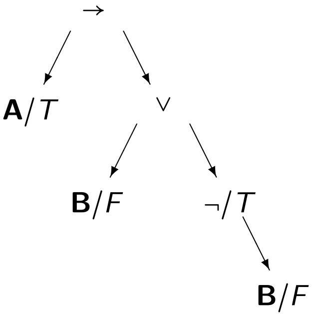

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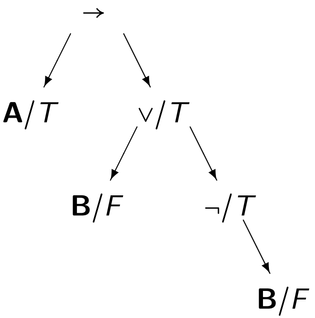

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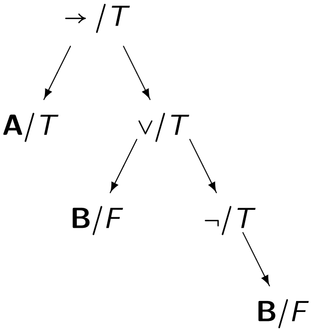
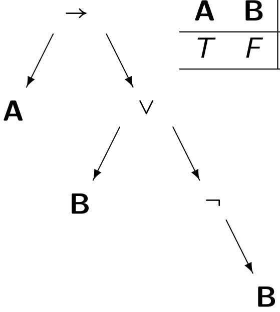


Truth assignment: $v(\mathbf{A}) = T$; $v(\mathbf{B}) = F$
Work truth values up tree, using common semantics of $\neg, \wedge, \vee, \to, \leftrightarrow$.
Say v satisfies τ, since the root of the tree is T.

Truth Table—One Row

Consider $(\mathbf{A} \rightarrow (\mathbf{B} \vee (\neg \mathbf{B})))$.



$\mathbf{A}$	$\mathbf{B}$	(	$\mathbf{A}$	$\rightarrow$	(	$\mathbf{B}$	$\vee$	(	$\neg$	$\mathbf{B}$	)	)	)
T	F												

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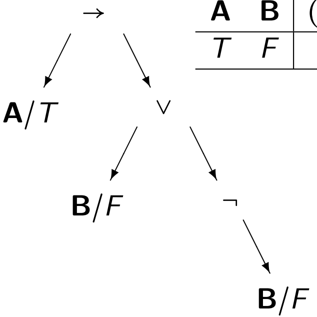

$\mathbf{A}$	$\mathbf{B}$	(	$\mathbf{A}$	$\to$	(	$\mathbf{B}$	$\vee$	(	$\neg$	$\mathbf{B}$	)	)	)
T	F		T			F				F			

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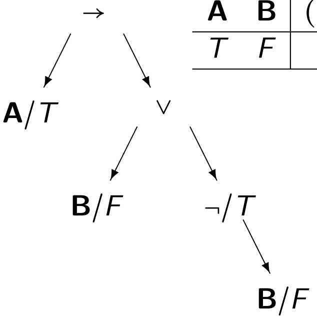

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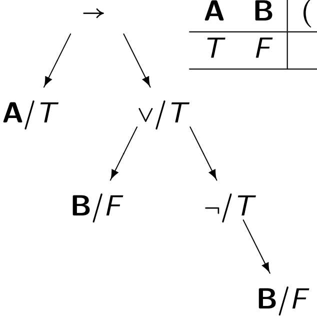

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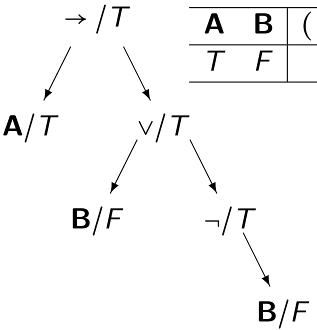
Consider $(\mathbf{A} \to (\mathbf{B} \lor (\neg\mathbf{B})))$.



A	**B**	(	**A**	$\to$	(	**B**	$\lor$	(	$\neg$	**B**	)	)	)
T	F		T	T		F	T		T	F			

Truth Table (All Rows)

Consider $(\mathbf{A} \rightarrow (\mathbf{B} \vee (\neg \mathbf{B})))$.

$\mathbf{A}$	$\mathbf{B}$	$(\mathbf{A}$	$\rightarrow$	$(\mathbf{B}$	$\vee$	$(\neg$	$\mathbf{B})))$
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Wff is true for all assignments to relevant sentence symbols. A tautology.

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F	T	F	T	T	T	F	T
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$n > 1$

- List all assignments to $\mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$, recursively, and also put $v(\mathbf{A}_n) = T$ in each assignment; then
- again list all assignments $\mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$, recursively, but this time also put $v(\mathbf{A}_n) = F$ in each assignment.

Inductively, there are 2^n.

A	**B**
T	T
F	T
T	F
F	F

Tautology and Truth Table Example

Give the truth table (all rows) for $(\neg(\mathbf{A} \wedge \mathbf{B})) \leftrightarrow ((\neg\mathbf{A}) \vee (\neg\mathbf{B}))$. How does one confirm that this is a tautology?

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A	**B**	(	$\neg$(**A**	$\wedge$	**B**))	$\leftrightarrow$	((	$\neg$**A**)	$\vee$	(	$\neg$**B**))
T	*T*		*F*	*T*		*T*		*F*	*F*		*F*
F	*T*		*T*	*F*		*T*		*T*	*T*		*F*
T	*F*		*T*	*F*		*T*		*F*	*T*		*T*
F	*F*		*T*	*F*		*T*		*T*	*T*		*T*

The "top level" WFF is T under all truth assignments to the sentence symbols.

Note on Implication

α	β	(	α	$\to$	β	)
T	T			T		
F	T			T		
T	F			F		
F	F			T		

This is our (common) semantics for $\to$. (No objections allowed.)

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0. If $A \in S$, then $\overline{v}(A) = v(A)$.
1. $\overline{v}((\alpha \wedge \beta)) = \begin{cases} T, & \text{if } \overline{v}(\alpha) = \overline{v}(\beta) = T; \\ F, & \text{otherwise.} \end{cases}$

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We say v satisfies φ if $\overline{v}(\varphi) = T$.

We say φ is satisfiable if some v satisfies φ.

All definitions

α	β	$(\neg\alpha)$	$(\alpha \wedge \beta)$	$(\alpha \vee \beta)$	$(\alpha \to \beta)$	$(\alpha \leftrightarrow \beta)$
T	T	F	T	T	T	T
F	T	T	F	T	T	F
T	F		F	T	F	F
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All is well (for our parenthesized WFFs), but proof will come later.

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Note: $\vDash \varphi$ iff $\neg\varphi$ is not satsifiable.

Tautology Examples

De Morgan:

$$
\begin{array}{rcl}
(\neg(\mathbf{A} \wedge \mathbf{B})) & \leftrightarrow & ((\neg \mathbf{A}) \vee (\neg \mathbf{B})) \\
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- Since $(\alpha \vee \beta) \vDash\Dashv (\neg((\neg\alpha) \wedge (\neg\beta)))$, we don't need $\vee$ if we have $\wedge$ and $\neg$.

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Note:

- Since $(\alpha \vee \beta) \vDash\Dashv (\neg((\neg\alpha) \wedge (\neg\beta)))$, we don't need $\vee$ if we have $\wedge$ and $\neg$.
- There's a duality between $\vee$ and $\wedge$.

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Contrapositive:

$$((\mathbf{A} \to \mathbf{B}) \leftrightarrow ((\neg \mathbf{B}) \to (\neg \mathbf{A}))).$$

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Double negative

$$(\mathbf{A} \leftrightarrow (\neg(\neg \mathbf{A}))).$$

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See http://en.wikipedia.org/wiki/Sidney_Morgenbesser for more Morgenbesserisms.

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The main concept of “truth” for sentential logic is tautological implication, $\Sigma \models \tau$.

Is τ true under any semantics (assignment) making Σ true?

For finite Σ, we can enumerate truth assignments to relevant sentence symbols and use the truth table method to check tautological implication.

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- Finite $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. Then $\Sigma \vDash \tau$ iff $\vDash \sigma_1 \wedge \cdots \wedge \sigma_n \to \tau$.
- Suppose, for all i, either $\mathbf{A}_i \in \Sigma$ or $\neg\mathbf{A}_i \in \Sigma$, decidably. Then $\Sigma \vDash \tau$ iff v satisfies τ for the truth assignment v induced by Σ. Showing $\Sigma \vDash \tau$ is showing this one v satisfies τ.

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- Tautologies, $\Sigma = \varnothing$. Since τ is finite, check $\vDash \tau$ using truth tables. Proving $\vDash \tau$ amounts to finding a tautology—doing logic, not math.
- Finite $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. Then $\Sigma \vDash \tau$ iff $\vDash \sigma_1 \wedge \cdots \wedge \sigma_n \to \tau$.
- Suppose, for all i, either $\mathbf{A}_i \in \Sigma$ or $\neg\mathbf{A}_i \in \Sigma$, decidably. Then $\Sigma \vDash \tau$ iff v satisfies τ for the truth assignment v induced by Σ. Showing $\Sigma \vDash \tau$ is showing this one v satisfies τ.

The interesting case: Infinite Σ. Several v's satisfy all $\sigma \in \Sigma$ and several v's do not. Showing $\Sigma \vDash \tau$ says something (new?) about a large set of v's.

We Seek the Truth

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The interesting case: Infinite Σ. Several v's satisfy all $\sigma \in \Sigma$ and several v's do not. Showing $\Sigma \models \tau$ says something (new?) about a large set of v's. What is true in the theory of Σ?

Example—Multiplication

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Σ can "say" that the above multiplication computation is correct, e.g., $\mathbf{A}_2 \leftrightarrow \mathbf{A}_0 \wedge \mathbf{A}_1$, etc, for any factors. (Σ is infinite.)

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Outline

Efficiency

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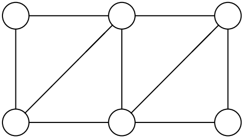
- Major open problem—P v. NP problem.
- Probably need time 2^n.
- Clay \$1M problem.
- At least as hard as thousands of other common problems with no efficient algorithm. (So many smart people have (implicitly) tried and failed to solve this problem efficiently.)

An NP Problem

GRAPH 3-COLORABILITY: Given a graph $G = (V, E)$, does there exist a coloring of vertices by red, blue, and maize, with no monochormatic edge?



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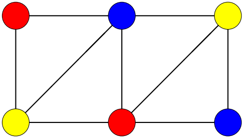

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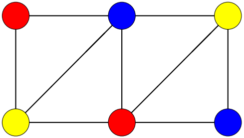

I.e., can we assign everyone to three committees, respecting stated conflicts?

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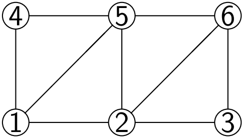

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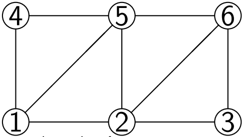

Let $C(i, j)$ (a sentence symbol) represent that vertex i is colored j. Then

$$L(1) = C(1, \text{red}) \vee C(1, \text{blue}) \vee C(1, \text{maize})$$

says that vertex 1 gets at least one color.

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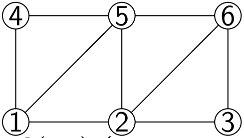

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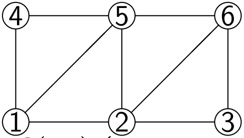

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$$E(1, 2, \text{red}) = \neg(C(1, \text{red}) \wedge C(2, \text{red}))$$

says that vertices 1 and 2 are not both colored red. (Repeat for each edge and each color.)

Punchline

Theorem

Let φ be the AND of all the constraints (at least one color per vertex, at most one color per vertex, no monochromatic edges.)
Then G is 3-colorable iff φ is satisfiable iff $\neg\varphi$ is not a tautology.

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- So if we could detect tautologies (equivalently, determine satisfiability) quickly, we could detect 3-colorability quickly. (Converse is also true!)
- No one knows how to solve the 3-colorability problem quickly.

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Learn secret key, one bit at a time. Trillions of dollars are betting no.

Outline

Procedures without proofs

We have procedures for most of the relevant tasks, but no proofs yet.

Done

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- Express basic English(?) concepts in Boolean logic (colorability, multiplication, ...)

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- "Semidecide" $\Sigma \vDash \tau$ for infinite Σ.
  - Given (a description of) Σ, rattle off all the τ for which $\Sigma \vDash \tau$.
  - We may not be able to decide, given Σ and τ, whether $\Sigma \vDash \tau$.