Math 481 Homework 3

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Due Wednesday, October 3, 2012, at start of class

(5 pts.; Based on Enderton 1.5.5.) Show that $\{\neg, \Leftrightarrow\}$ is not complete. Express the exclusive or function $+$, as well as $\top$ (constant true) and $\bot$ (constant false) using just $\{\neg, \Leftrightarrow\}$, and conclude that $\{\top, \bot, \neg, \Leftrightarrow, +\}$ is not complete, either. *Suggestion: a wff using these connectives and only* $\mathbf{A}$ *and* $\mathbf{B}$ *is satisfied by an even number of truth assignments.*

In fact, any wff on n sentence symbols using these connectives is satisfied by either $0, 2^n$, or 2^{n-1} truth assignments, *i.e.*, none, all, or half. If you're bored with the above, try this. *Suggestion: This can be shown using linear algebra. Use the fact that, if x and y are vectors over $\mathbf{Z}_2$, then, for fixed y, we have $\langle x, y\rangle = 0$ for half or all of the x's and $\langle x, y\rangle = 1$ for the remaining half or none of the x's, respectively. Relate x to a truth assignment and y to a wff, and use our induction principle.*

(5 pts.) A Boolean function is *monotonic* if whenever we fix all but one input and change the remaining input from false to true, the output of the function does not change from true to false. Show that $\{\wedge, \vee\}$ is complete for the monotonic functions.

(5 pts.) A Conjunctive Normal Form is a wff of the form $\alpha_1 \wedge \alpha_2 \wedge \cdots$ (finite length, as usual), where each α_i is of the form $\beta_1 \vee \beta_2 \vee \cdots$ (again, finite length), and β is a *literal*: either a sentence symbol or a negation. For example,

$$(\mathbf{A}_1 \vee \mathbf{A}_3 \vee \neg \mathbf{A}_4) \wedge (\mathbf{A}_1 \vee \neg \mathbf{A}2)$$

is in CNF.

1. Show that, for any wff ϕ, there is an equivalent wff ψ in conjunctive normal form. (Suggestion: Think about DNFs, too.)

2. Consider $\phi = \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_n$, where $+$ denotes the exclusive or. Show that any tautologically equivalent CNF has size exponential in n, *i.e.*, for some $c > 0$, the size is at least 2^{cn}.

3. Show how, given any wff ϕ, to construct a CNF ψ such that ψ is satisfiable iff ϕ is satisfiable and the size of ψ is linear in the size of of ϕ, *i.e.*, for some c, if ϕ has n connectives, then ψ has at most cn connectives.