Compactness and Effectiveness

Outline

Infinite?

"Given" an infinite set Σ of wffs and a single wff τ, can we "tell" whether $\Sigma \models \tau$?

Four Big Results

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- There is a set S that is r.e. but not decidable (i.e., no computer that halts on all inputs can, given x, determine whether $x \in S$).
- There is a decidable set Σ such that $\overline{\Sigma} = \{\tau : \Sigma \models \tau\}$ is not decidable.

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If $\Sigma \vDash \tau_i$, then, by compactness, some $\{\sigma_1, \sigma_2, \ldots, \sigma_j\} \vDash \tau_i$. So we don't miss any consequences.

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- $\Sigma \vDash \mathbf{A}_n$ iff $\exists t\ M$ prints n in t steps—undecidable.

Significance

We can not decide which wffs are the consequences of some "scenario" specified by Σ, but we can enumerate those consequences (analogs of "theorems" in first-order logic).
This narrowly characterizes the "wildness" of $\{\tau : \Sigma \models \tau\}$.
(Additional comments later.)

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The interesting case: Infinite Σ. Several v's satisfy all $\sigma \in \Sigma$ and several v's do not. Showing $\Sigma \models \tau$ says something (new?) about a large set of v's. What is true in the theory of Σ?

Example—Multiplication

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\cdots & \mathbf{A}_5 & \mathbf{A}_1 \\
\hline
\cdots & \mathbf{A}_6 & \mathbf{A}_2 \\
\cdots & \mathbf{A}_7 & \\
\cdots & & \\
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This is true under any assigment to the sentence symbols making Σ correct, and interesting.

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Here, one can show that $\mathbf{A}_8$ depends only on $\mathbf{A}_{<8}$, *i.e.*, $\mathbf{A}_2, \mathbf{A}_6, \mathbf{A}_7$ are correct functions of $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_4, \mathbf{A}_5$.

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There is no effective procedure, in general, to find Σ_0.

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Suppose $\tau = \mathbf{A}_1$ and

$$\Sigma = \{\mathbf{A}_2, \mathbf{A}_2 \to \mathbf{A}_3 \wedge \mathbf{A}_4, \mathbf{A}_3 \to \mathbf{A}_{16} \wedge \mathbf{A}_{13}, \ldots\}.$$

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Insufficient to restrict attention to WFFs in Σ that mention $\mathbf{A}_1$.

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Interesting direction: If Σ is finitely satisfiable, then it is satisfiable.
Proof has two parts.

- Enlarge Σ to Δ, a maximal such set.
- Show that $v(\phi) = T$ iff $\phi \in \Delta$ works.

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Let $\Delta = \bigcup_i \Delta_i$. Then:

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- All $\Delta_{>0}$ are finitely satisfiable. (Inductive case, next...)
- Δ is finitely satisfiable (any finite $\Delta' \subseteq \Delta$ is also a subset of some Δ_i).

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- Toward a contradiction, assume Δ' is a finite, unsatisfiable subset of $\Delta_{i+1} = \Delta_i; \neg\alpha_i$. Similar to the above, $\neg\alpha_i \in \Delta'$.

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- Otherwise, there's there's some finite, unsatisfiable $\Delta'' \subseteq \Delta_i; \alpha_i$. Since Δ_i is finitely sastisfiable, $\Delta'' \nsubseteq \Delta_i$, so $\alpha_i \in \Delta''$ by induction.
- Toward a contradiction, assume Δ' is a finite, unsatisfiable subset of $\Delta_{i+1} = \Delta_i; \neg\alpha_i$. Similar to the above, $\neg\alpha_i \in \Delta'$.
- $\widehat{\Delta} = (\Delta' - \neg\alpha_i) \cup (\Delta'' - \alpha_i) \subseteq \Delta_i$ is satisfiable by induction, by some $\overline{v}$.

All Δ_i are finitely satisfiable

$$\Delta_{i+1} = \begin{cases} \Delta_i;\alpha_i & \text{if this is finitely satisfiable} \\ \Delta_i;\neg\alpha_i & \text{otherwise} \end{cases}$$

To show Δ_{i+1} is finitely satisfiable:

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- If $\overline{v}(\alpha_i) = T$, then $\overline{v}$ satisfies $\widehat{\Delta};\alpha_i \supseteq \Delta''$. Else $\overline{v}(\neg\alpha_i) = T$, and $\overline{v}$ satisfies $\widehat{\Delta};\neg\alpha_i \supseteq \Delta'$.

Defining a Satisfying Assignment

Define $v(\mathbf{A}) = T$ iff $\mathbf{A} \in \Delta$ and extend to $\overline{v}$.

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In particular, $\overline{v}$ satisfies Σ.

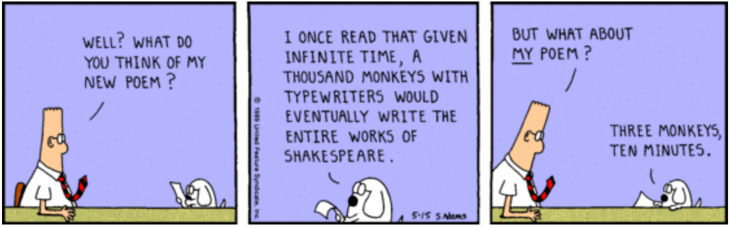
Corollary

An equivalent formulation:
$\Sigma \models \tau$ iff there is a finite $\Sigma_0 \subseteq \Sigma$ with $\Sigma_0 \models \tau$.

Outline

Enumeration

http://dilbert.com



Recursive Enumerability

A set S (of numbers, expressions, or WFFs) is recursively enumerable (r.e.) if there is some computer program (machine) M that prints a (possibly infinite) list that contains exactly the elements of $S = L(M)$.

Recursive Enumerability

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A set is semi-decidable if there is a computer program that, on input x, halts (and answers "yes") or does not halt depending on whether $x \in S$ or $x \notin S$, respectively.

Equivalence

Theorem

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Interleaving processes: Proceed one step of $M(x_0)$ at time 1, 3, 5, 7,. . . . Proceed one step of $M(x_1)$ at time 2, 6, 10, 14,. . . . Proceed one step of $M(x_i)$ at time odd$\cdot 2^i$. □

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- There are sets that are not decidable. (Most of the uncountably-many sets are not decidable by any of the countably-many computer programs.)
- There are sets that are r.e. but not decidable. (Later...)

Practical Analogy

TCP, a communications protocol underlying the web, requires:

- If a server is normal, it must send an acknowledgement of clients' packets.
- If a server is overloaded, it must signal this by not sending acknowledgements. (That may be all it can manage.)

(Im-)morally speaking, the set of times at which a server is normal is r.e. but not decidable.

Tautological Consequences is R.E.

Theorem

If Σ is r.e., then so is the set $\overline{\Sigma}$ of its tautological consequences.

Proof.

- Let $\Sigma = \langle \sigma_1, \sigma_2, \ldots, \rangle$ and WFF $= \langle \tau_1, \tau_2, \ldots, \rangle$.

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How to try all? Next...
By compactness, some $\{\sigma_1, \sigma_2, \ldots, \sigma_j\} \vDash \tau_i$, iff $\Sigma \vDash \tau_i$. $\square$

Busy Beaver

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Enumerate all pairs of natural numbers.

$f(i,j)$	1	2	3	4
0	0	1	3	6
1	2	4	7	
2	5	8		

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Interleave all processes: advance $f(i,j)$ one step at time odd$\cdot 2^{f(i,j)}$.

$f(i,j)$	Times when $f(i,j)$ is active
0	$1, 3, 5, 7, 9, 11, 13, 15, \ldots$
1	$2, 6, 10, 14, \ldots$
2	$4, 12, \ldots$
3	$8, \ldots$

Outline

The Reals are Uncountable

Suppose not. List the reals:

Position	Real
1	.12345...
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x cannot be on the list.

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- Define k'th digit of x to be 0 or 9 so that x is at least $2\cdot 10^{-k}$ away from any root of the k'th polynomial, when k'th digit is chosen. Here, $x = 0.9000\ldots$

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- However the lower-order bits are set, x is not the root of the k'th polynomial; it is at least 10^{-k} away.

Undecidability of the Halting and Related Problems

The Acceptance problem is given by

$$A = \{(M, x) : M(x) \text{ halts and } M(x) = \text{yes}\}$$

i.e., the set of (M, x) such that computer program ("machine") M, when run on input x, halts and answers yes.

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E.g., a malware detector might want to decide A, by examining code for M without running M.
Suppose there were a decider H for A, i.e.,

$$H(M,x) = \begin{cases} \text{yes} & M(x) \text{ halts and } M(x) = \text{yes} \\ \text{no} & M(x) \text{ runs forever or } M(x) = \text{no} \end{cases}$$

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D halts on all inputs, since H does.

Punchline

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Contradiction! Thus decider H does not exist. $\boxed{A \text{ is not decidable.}}$

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- If M runs forever, then S_A runs forever.

Compare with Transcendentals

To produce transendental x, list integer polynomials and make sure the k'th polynomial p does not "accept" x, *i.e.*, $p(x) \neq 0$. To insure the condition for p_k, look at the k'th bit of x.

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To produce transendental x, list integer polynomials and make sure the k'th polynomial p does not "accept" x, *i.e.*, $p(x) \neq 0$. To insure the condition for p_k, look at the k'th bit of x.

To produce undecidable set A, list all machines and make sure machine M does not decide A, *i.e.*, there's some input y on which M does not halt or $M(y)$'s output doesn't match A. To insure the condition for machine M, look at the input $y = M$ for machine M.

Consequences are Undecidable

- Suppose S_A semi-decides $A = \{(M, x) : M(x) \text{ halts and } M(x) = \text{yes}\}$.

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- Let $\Sigma = \{\overbrace{\mathbf{A}_n \wedge \mathbf{A}_n \wedge \cdots \wedge \mathbf{A}_n}^{t} : (n, t) \in A_{\leq}\}$; then Σ is decidable.

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- Let $\Sigma = \{\overbrace{\mathbf{A}_n \wedge \mathbf{A}_n \wedge \cdots \wedge \mathbf{A}_n}^{t} : (n, t) \in A_{\leq}\}$; then Σ is decidable.
- $\Sigma \models \mathbf{A}_n$ iff $\exists t\ \overbrace{\mathbf{A}_n \wedge \mathbf{A}_n \wedge \cdots \wedge \mathbf{A}_n}^{t} \in \Sigma$ iff $\exists t\ \ S_A$ accepts (M, x) in t steps iff $(M, x) \in A$—undecidable!

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More is True...

We now show that there is a true but unprovable sentence in Boolean logic.

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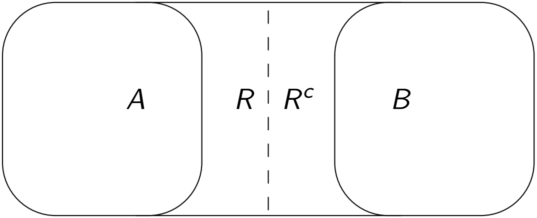
(More importantly, we formulate this carefully!)
First, a bit more recursion theory.

Recursive Inseparability

Two sets A and B are Recursively Inseparable if there is no decidable set R with $A \subseteq R$ and $B \subseteq R^c$.



Example Inseparable Sets

Theorem

The sets

$$\begin{aligned} L_Y &= \{M : M(M) \text{ halts and } M(M) = \text{yes}\} \\ L_N &= \{M : M(M) \text{ halts and } M(M) = \text{no}\} \end{aligned}$$

are recursively inseparable.

Proof.

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$$M(M) = \text{yes} \quad \Rightarrow \quad M \in L_\mathrm{Y}, \text{ by def of } L_\mathrm{Y}$$

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Suppose wlog $L_\mathrm{N} \subseteq R$ and $R \cap L_\mathrm{Y} = \varnothing$ with M deciding R. What is $M(M)$?

$$\begin{aligned} M(M) = \text{yes} &\Rightarrow M \in L_\mathrm{Y}, \text{ by def of } L_\mathrm{Y} \\ &\Rightarrow M \notin R, \text{ since } R \cap L_\mathrm{Y} = \varnothing \end{aligned}$$

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are recursively inseparable.

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Suppose wlog $L_\mathrm{N} \subseteq R$ and $R \cap L_\mathrm{Y} = \varnothing$ with M deciding R. What is $M(M)$?

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Example Inseparable Sets

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The sets

$$\begin{aligned} L_Y &= \{M : M(M) \text{ halts and } M(M) = yes\} \\ L_N &= \{M : M(M) \text{ halts and } M(M) = no\} \end{aligned}$$

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Contradiction! $\boxed{\text{No recursive separating } R.}$ □

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Also, note $M \in L_i$ iff $\Sigma_{i,\le} \models \mathbf{A}_M$, $i = $Y,N.

Inseparable

Note $M \in L_i$ iff $\Sigma_{i,\leq} \vDash \mathbf{A}_M$, i =Y,N.

Theorem

$\overline{\Sigma_{Y,\leq}}$ and $\overline{\Sigma_{N,\leq}}$ are recursively inseparable.

Proof.

If some recursive R' separated $\overline{\Sigma_{\mathrm{Y},\leq}}$ and $\overline{\Sigma_{\mathrm{N},\leq}}$, then $R = \{M : \mathbf{A}_M \in R'\}$ would separate L_{Y} and L_{N}. $\square$

Significance

Theorem

Suppose

- *v is any truth assignment such that $\overline{v}$ satisfies every $\sigma_Y \in \Sigma_{Y,\leq}$ and satisfies no $\sigma_N \in \Sigma_{N,\leq}$; equivalently, $\overline{v}$ satisfies $\Sigma_{\leq} = \Sigma_{Y,\leq} \cup \{\neg\sigma : \sigma \in \Sigma_{N,\leq}\}$ (v is any reasonable notion of truth compatible with $\Sigma_{\leq}$), and*

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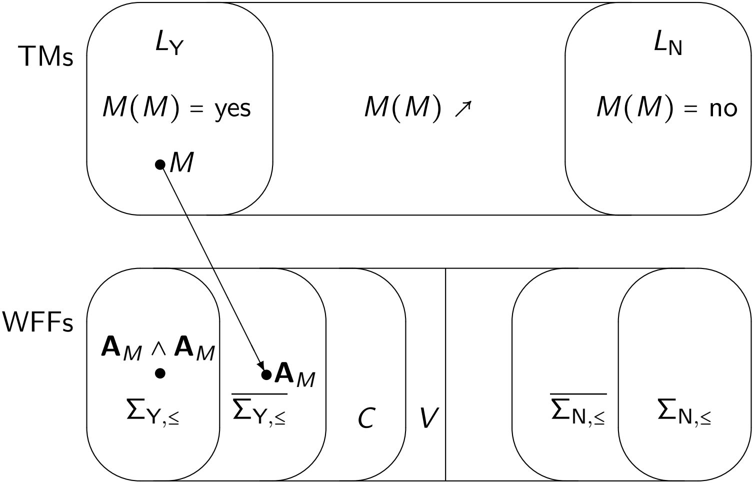
Proof.

- $V = \{\sigma : \overline{v}(\sigma) = T\}$ is not decidable, so $C \neq V$.
- If $V \subseteq C$, then $\exists\sigma \in V$ with $\sigma \in C$ and $\neg\sigma \in C$.
- Since C is closed, $C =$ WFF. (Contradiction.)

Picture


TMs
L_{Y}
$M(M)$ = yes
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$\mathbf{A}_M \wedge \mathbf{A}_M$
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Some [sic] notion of truth?

Some (different) examples:

1. $v(\mathbf{A}_M)$ is true iff $M(M)$ halts and says yes.
2. $v(\mathbf{A}_M)$ is true unless $M(M)$ halts and says no.
3. $v(\mathbf{A}_M)$ is true iff $M(M)$ halts and says yes, or if $M(M)$ loops and $|M|$ is a prime number.

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Later, in first-order logic, the definition of $\mathbf{A}_M$ becomes part of the logic.

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Reformulation:
For certain Σ, given any v compatible with Σ and any r.e., closed, not-all-WFF proof system C, there is a true but unprovable sentence.