Semantics of First-order Well-formed Formulae

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1 Semantics

Recall our grammar for wffs:
wff: atomic-formula $\big|$ wff $\to$ wff $\big|$ $\neg$ wff $\big|$ $\forall$ variable wff
atomic-formula: Predicate term term term ...
term: constant $\big|$ variable $\big|$ function term term term ...
constant: $a\big|b\big|c\big|\cdots$
variable: $x\big|y\big|z\big|\cdots$
function: $f\big|g\big|h\big|\cdots$
predicate: $P\big|Q\big|R\big|\cdots$

Here function symbols and predicate symbols must have the correct arity, which we take to be a syntactic property of the symbol.

The goal now is to give a recursive definition for $\models_{\mathfrak{A}} \phi[s]$, "$\phi$ is true in $\mathfrak{A}$ with s," where $\mathfrak{A}$ is a structure, giving

- a universe, $|\mathfrak{A}|$, where variables range
- an interpretation, $f^{\mathfrak{A}} : |\mathfrak{A}|^n \to |\mathfrak{A}|$, for each n-ary function symbol, f (including 0-ary functions, the constants)
- an interpretation, $P^{\mathfrak{A}} : |\mathfrak{A}|^n \to \{T, F\}$, for each n-ary predicate symbol, P;

and s is an assignment to variables of elements of the universe, $|\mathfrak{A}|$.

2 Examples and Exercises

2.1 Bad Apples

Define the semantics of the predicate symbols A and B so that Ax means "x is an apple" and Bx means "x is bad." Then:

- Using $\forall$, express "All apples are bad."
- Using $\exists$, express "Some apple is bad."
- What does $\exists x\ Ax \to Bx$ mean?

2.2 Lincoln

In honor of Daniel Day Lewis's film, give several possible formalisms for:

- You can fool some of the people all of the time.
- You can fool all of the people some of the time.
- You can not fool all of the people all of the time.

2.3 Graph Theory

In the language of graph theory, there is equality and one binary predicate symbol, E, for "edge," written Exy or $x \sim y$ for "x is adjacent to y." For each of the following wffs, give a structure, $\mathfrak{A}$, and variable assignment, s, making the wff true and another $\mathfrak{A}$ and s making the wff false, such that $\forall x \forall y x \sim y \to y \sim x$ in your structure. Explain in English the set of all structures and assignments making the wff true.

- $\forall x\, x \sim y$
- $\forall x \exists y\, x \sim y$
- $\exists y \forall x\, x \sim y$
- $\forall x \forall z \exists y\, x \sim y \wedge y \sim z$.
- $(\forall x \forall y \forall z\, Cx = Cy \vee Cx = Cz \vee Cy = Cz) \wedge (\forall x \forall y\, x \sim y \to Cx \neq Cy)$, where C is an additional unary function.

3 Sentences

If ϕ has no free variables, it is called a *sentence*. In that case, s is irrelevant. (This is proved recursively, and variable assignments actually are needed in the innards of the proof. See Enderton, pages 86–87.) We write $\models_{\mathfrak{A}} \phi$. (Do if time on November 14.) If ϕ is a sentence and $\models_{\mathfrak{A}} \phi$, we say that $\mathfrak{A}$ *models* ϕ.

4 Logical Implication

Write $\Gamma \models \phi$ if Γ is a (possibly infinite) set of wffs, ϕ is a wff, and whenever a structure $\mathfrak{A}$ and variable assignment s satisfy every $\gamma \in \Gamma$, $\mathfrak{A}$ and s also satisfy τ.

Examples.

- $\forall x Qx \models Qy$ (This and the next example answers a question of Justin Dimmel on November 12.)
- $Qy \not\models \forall x Qx$.

- In the language of graph theory, $\forall x \forall y x \sim y \to y \sim x \models \phi$ if ϕ is true of all undirected graphs, though not necessarily in directed graphs.

Peek ahead: We will shortly give a symbol-pushing deductive system, and write $\Gamma \vdash \phi$ if there's a deduction from Γ to ϕ. A deduction is a sequence of wffs, ending in ϕ, such that each wff is either in Γ or a logical axiom such as a tautology or something like $\forall x Qx \to Qy$, or follows by modus ponens; ψ is allowed to follow ϕ and $\phi \to \psi$. We also write $\models \phi$ (and say "ϕ is valid") if ϕ is true with every structure and variable assignment. For example, $\models \forall x Qx \to Qy$.

We now have four possible notions related to truth of ϕ, none fully acceptable:

- $\models_{\mathfrak{A}} \phi$. ϕ is true in the particular structure $\mathfrak{A}$. In a sense, this is what we, as working mathematicians, want to formalize and to study. Logic provides tools for formalizing this (see next). If we simply try, however, to argue about $\models_{\mathfrak{A}} \phi$ directly, we would be avoiding logic and rigor.

- $\Gamma \models \phi$. ϕ is true whenever everything in Γ is true. The idea would be to pick Γ carefully to isolate $\mathfrak{A}$, then pound on $\Gamma \models \phi$ to determine whether $\models_{\mathfrak{A}} \phi$. Usually we can pick Γ so that that $\mathfrak{A}$ makes everything in Γ true, but it is harder to pick Γ so that $\mathfrak{A}$ is the only model for Γ. In interesting cases, it is inherently impossible to pick Γ that picks out $\mathfrak{A}$ uniquely.

- $\Gamma \vdash \phi$. Here we can enumerate $\{\phi : \Gamma \vdash \phi\}$, provided we can enumerate Γ (usually required). This is *a priori* much easier than checking $\Gamma \models \phi$, since the latter involves trying all possible universes for $|\mathfrak{A}|$, etc. Remarkably, it turns out that $\Gamma \models \phi$ iff $\Gamma \vdash \phi$ for the deductive system $\vdash$ that we will shortly give. (Also for other $\vdash$'s.) If Γ is recursively enumerable, then so is $\{\phi : \Gamma \vdash \phi\}$. Thus $\Gamma \vdash \phi$ is an improvement over $\Gamma \models \phi$ in terms of our ability to analyze, but it still suffers from the fact that often we can't have Γ pick out $\mathfrak{A}$ uniquely and $\{\phi : \Gamma \vdash \phi\}$ is not decidable.

- $\models \phi$. Here ϕ is true by general abstract nonsense, and is not capturing anything about any working mathematical object of interest. So ϕ is true in the universe of numbers, people, colors, etc. It does capture a correct logical argument, like syllogism. Still, the relatively simple set $\{\phi : \models \phi\}$ of validities is not decidable; by comparison, recall that the set of (boolean) tautologies is decidable, by the truth table method.

In fact, the tue sentences of number theory and the negations of validities (*i.e.*, $\neg\sigma$ when $\models \sigma$) are recursively inseparable: There is no computer that

- says YES on (even a smallish subset of) the true sentences over $(0, 1, +, \cdot, <)$, like $\forall x \exists y\, 2 \cdot y = x \vee 2 \cdot y + 1 = x$;

- says NO on negations of validities, like $\neg \forall x x = x$, and

- is allowed to answer YES or NO arbitrarily (but must halt) on other wffs.

5 Expressibility

We know that certain concepts can be expressed in first order. For example, in the language of graphs, the set of vertices at distance 2 or less from a given (constant) vertex c can be expressed as $c = x \vee c \sim x \vee \exists y\, c \sim y \wedge y \sim x$. Note that x is free here. In general, we can ask whether a subset of the universe (or a set of pairs, etc., over the universe), can be expressed in first-order logic.

Also interesting is showing that certain concepts can *not* be expressed...