Math 53: Lec 002 Sec 213

The Fundamental Theorem for Line Integrals

1 November 2012

Learning Objectives: 1) What is a line integral over a vector field? 2) How to calculate line integrals over a vector field? 3) What is the Fundamental Theorem for Line Integrals?; 4) For what kind of vector fields does the fundamental theorem apply?; 5 How can you test whether a direction field is conservative?

I. Reading Quiz

1. How do you define W work as a line integral?

2. Write the formula for a line integral over a vector field $\mathbf{F}$. There are several formulas for this. What is $\mathbf{T}$?

3. **True/False**: $\int_{-C} \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ because line integral with respect to arc length is independent of direction.

4. Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the quarter circle $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \le t \le \pi/2$.

5. If $\mathbf{F}$ is given in components, i.e. $\mathbf{F}$=$P\mathbf{i}$+$Q\mathbf{j}$+$R\mathbf{j}$ how would your evaluate your line integral?

6. State the Fundamental Theorem for Line Integrals. What is this theorem helpful for?

7. Find the potential function for $\mathbf{F}(x, y) = (3x + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$, and evaluate the line integral where C is the curve given by $\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j}$. $0 \le t \le \pi$.

8. What is a conservative vector field and how do you determine if a vector field is conservative?

9. What does it mean for a line integral to be path independent? How can you check?

10. What is the relationship between path independence, conservative vector field and the fundamental theorem?

II. Problems

NORMS & INSTRUCTIONS: 1) Take 5 minutes to read over the problems and think about how you would approach it; 2) Be in a group of 4 students maximum

1. Let $\mathbf{F}(x, y) = (y^2 + 1)\mathbf{i} + (2xy - 2)\mathbf{j}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where
(a) C is a straight-line path from $(0, 0)$ to $(1, 1)$.
(b) C is a path from $(0, 0)$ to $(1, 1)$ along the parabola $y = x^2$.

2. Suppose $\mathbf{F}$ is the gradient of some function. What is the work done by $\mathbf{F}$ along a closed curve (i.e. a curve that comes back to where it started)?

3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = (2x^3y^4 + x)\mathbf{i} + (2x^4y^3 + y)\mathbf{j}$ and C is given by $\mathbf{r}(t) = (t\cos(\pi t) - 1)\mathbf{i} + \sin\left(\frac{\pi t}{2}\right)\mathbf{j}$, $0 \leq t \leq 1$. If you want to use the Fundamental Theorem, be sure to show that it is conservative.

4. Which of the following are conservative vector fields? For each conservative vector field, find a function for which it is a gradient.
(a) $\mathbf{F}(x, y) = (2xy + y)\mathbf{i} + (x^2 + x)\mathbf{j}$
(b) $\mathbf{F}(x, y) = \sin(xy)\mathbf{i} + \mathbf{j}$
(c) $\mathbf{F}(x, y) = y^2\mathbf{i} + (2xy + x)\mathbf{j}$
(d) $\mathbf{F}(x, y) = x^2\mathbf{i} + 3xz\mathbf{k}$