MDPs: Bellman Equations, Value Iteration

Sutton & Barto Ch 4 (Cf. AIMA Ch 17, Section 2-3)

Adapted from slides kindly shared by Stuart Russell

Appreciations

◇ My older brother Rusty and his late wife Debbie

◇ Online community

Share some of yours?

Announcements

Reminder: Project P2 Multi-Agent Pac-Man is out, due Thu Nov 1

Guest lecture next Monday

Outline

- ♢ Dynamic programming and Bellman equations
- ♢ Value iteration
- ♢ Policy iteration

Credit to Dan Klein, Stuart Russell and Andrew Moore for most of today's slides

Books, Notation

◇ AIMA: Artifical Intelligence, a Modern Approach, by Russell & Norvig

Less detail on value iteration, reinforcement learning, etc.

Nice graphs

◇ Reinforcement Learning, An Introduction, by Sutton & Barto

More coverage, useful for this part of the course: chapters 3, 4, and 6

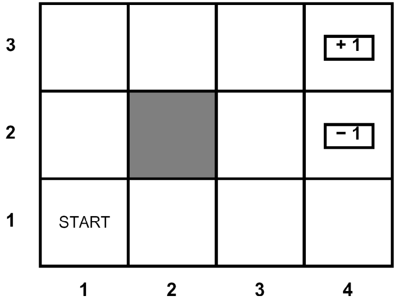
Uses same nomenclature we'll use in lecture for optimal values / utilities

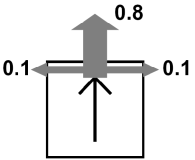
"V*(s)" and "Q*(s)" vs "U(s)"

Cool mutual recursion

Grid World

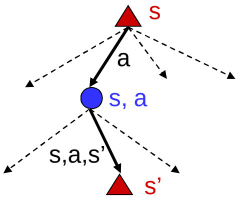
- The agent lives in a grid
- Walls block the agent's path
- The agent's actions do not always go as planned:
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- Small "living" reward each step
- Big rewards come at the end
- Goal: maximize sum of rewards*





Recap: MDPs

- Markov decision processes:
  - States S
  - Actions A
  - Transitions P(s'|s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount γ)
  - Start state s_0

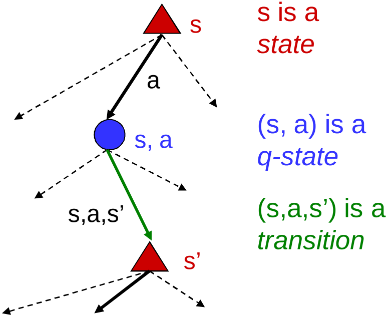


- Quantities:
  - Policy = map of states to actions
  - Episode = one run of an MDP
  - Utility = sum of discounted rewards
  - Values = expected future utility from a state
  - Q-Values = expected future utility from a q-state

[DEMO – MDP Quantities]

Optimal Utilities

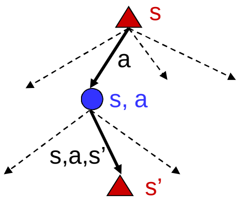
- The utility of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally
- The utility of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
  $\pi^*(s)$ = optimal action from state s



Bellman Equations

- Definition of utility leads to a simple one-step lookahead relationship amongst optimal utility values:

  Total optimal rewards = maximize over choice of (first action plus optimal future)



- Formally:

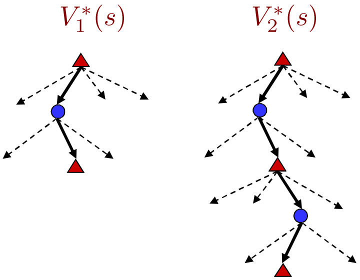
$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

Value Estimates

- Calculate estimates $V_k^*(s)$
  - Not the optimal value of s!
  - The optimal value considering only next k time steps (k rewards)
  - What you'd get with depth-k expectimax*
  - As k $\rightarrow \infty$, it approaches the optimal value*



- Almost solution: recursion (i.e. expectimax)
- Correct solution: dynamic programming

[DEMO -- V_k]

Value Iteration

- Idea:
  - Start with $V_0^*(s) = 0$ for all s, which we know is right (why?)
  - Given V_i^*, calculate the values for all states for depth i+1:

$$V_{i+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i(s') \right]$$

  - Throw out old vector V_i^*
  - Repeat until convergence
  - This is called a value update or Bellman update

- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

Example: γ=0.9, living reward=0, noise=0.2

Example: Bellman Updates

V_1

	1	2	3	4
3	0	0	0 →	+1
2	0		0	−1
1	0	0	0	0

V_2

	1	2	3	4
3	0	0	0.72	+1
2	0		0	−1
1	0	0	0	0

$$V_{i+1}(s) = \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma\, V_i(s')\right]$$

$$V_2(\langle 3,3\rangle) = \sum_{s'} T(\langle 3,3\rangle, \text{right}, s')\left[R(\langle 3,3\rangle) + 0.9\, V_1(s')\right]$$

max happens for a=right, other actions not shown

$$= 0.9\,[0.8 \cdot 1 + 0.1 \cdot 0 + 0.1 \cdot 0]$$

Example: Value Iteration

V_2

	1	2	3	4
3	0	0	0.72	+1
2	0		0	−1
1	0	0	0	0

V_3

	1	2	3	4
3	0	0.52	0.78	+1
2	0		0.43	−1
1	0	0	0	0

- Information propagates outward from terminal states and eventually all states have correct value estimates

Convergence*

- Define the max-norm: $||U|| = \max_s |U(s)|$
- Theorem: For any two approximations U and V

$$||U^{t+1} - V^{t+1}|| \leq \gamma \, ||U^t - V^t||$$

  - I.e. any distinct approximations must get closer to each other, so, in particular, any approximation must get closer to the true U and value iteration converges to a unique, stable, optimal solution
- Theorem:

$$||U^{t+1} - U^t|| < \epsilon, \Rightarrow ||U^{t+1} - U|| < 2\epsilon\gamma/(1-\gamma)$$

  - I.e. once the change in our approximation is small, it must also be close to correct

Practice: Computing Actions

- Which action should we chose from state s:
  - Given optimal values V?

$$\arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

  - Given optimal q-values Q?

$$\arg\max_a Q^*(s,a)$$

  - Lesson: actions are easier to select from Q's!

[DEMO – MDP action selection]

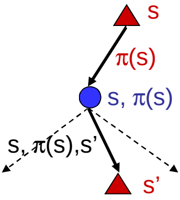
Utilities for a Fixed Policy

- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy

- Define the utility of a state s, under a fixed policy π:

  $V^\pi(s)$ = expected total discounted rewards (return) starting in s and following π

- Recursive relation (one-step look-ahead / Bellman equation):



$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^\pi(s')]$$

[DEMO – Right-Only Policy]

Policy Evaluation

- How do we calculate the V's for a fixed policy?
- Idea one: turn recursive equations into updates

$$V_0^\pi(s) = 0$$

$$V_{i+1}^\pi(s) \leftarrow \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V_i^\pi(s')]$$

- Idea two: it's just a linear system, solve with Matlab (or whatever)

Policy Iteration

- Alternative approach for optimal values:
  - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
  - Step 2: Policy improvement: update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges

- This is policy iteration
  - It's still optimal!
  - Can converge faster under some conditions

Policy Iteration

- Policy evaluation: with fixed current policy π, find values with simplified Bellman updates:
  - Iterate until values converge

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} T(s, \pi_k(s), s') \left[R(s, \pi_k(s), s') + \gamma V_i^{\pi_k}(s')\right]$$

- Policy improvement: with fixed utilities, find the best action according to one-step look-ahead

$$\pi_{k+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_k}(s')\right]$$

Comparison

- Both VI and PI compute the same thing (optimal values for all states)
- In value iteration:
  - Every pass (or “backup”) updates both utilities (explicitly, based on current utilities) and policy (implicitly, based on current utilities)
  - Tracking the policy isn’t necessary; we take the max

$$V_{i+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i(s')\right]$$

- In policy iteration:
  - Several passes to update utilities with fixed policy
  - After policy is evaluated, a new policy is chosen
- Both are dynamic programs for solving MDPs

Asynchronous Value Iteration*

- In value iteration, we update every state in each iteration
- Actually, *any* sequences of Bellman updates will converge if every state is visited infinitely often
- In fact, we can update the policy as seldom or often as we like, and we will still converge
- Idea: Update states whose value we expect to change: If $|V_{i+1}(s) - V_i(s)|$ is large then update predecessors of s