

Week 2: §4.3–§4.6

Basics of Probability

1. An experiment consists of randomly choosing one letter from the word *computer*.

a) The set of all possible outcomes is $S = \{c, o, m, p, u, t, e, r\}$

Definition: The set S of all outcomes of an experiment is the **sample space** of that experiment. An **event** is a subset of a sample space.

b) Find the outcomes in the following events.

i) The event that a vowel is chosen: $W = \{o, u, e\}$

ii) The event that a consonant is chosen: $X = \{c, m, p, t, r\}$

iii) The event that p is chosen: $Y = \{p\}$

iv) The event that u is chosen: $Z = \{u\}$

Definition: A **simple event** is an event that contains exactly one outcome.

c) Which events above are simple? Y and Z

d) Can Y and Z occur at the same time? no

e) What is the event $Y \cap Z$? \emptyset

Remark: All simple events are mutually exclusive.

f) How many ways can Y occur? In other words, what is $n(Y)$? 1

g) What is $\Pr(Y)$? $\frac{1}{8}$

h) How many ways can Z occur? 1

i) What is $\Pr(Z)$? $\frac{1}{8}$

j) Is $\Pr(Y) = \Pr(Z)$? Why or why not? *Yes. The probability of choosing one letter is the same as that of choosing any other.*

Definition: A *uniform sample space* is a sample space in which all outcomes are equally likely.

k) Complete the following table with the correct probabilities.

Outcome:	C	O	M	P	U	T	E	R
Probability:	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

The sum of the values in the second row is 1.

Remark: The table above is an example of a *probability distribution table*. A probability distribution table always has the following properties:

- the events in the table are all mutually exclusive; and
 - the sum of all of the probabilities is 1.
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l) What outcomes are in $Y \cup Z$? $\{P, U\}$

m) What is $\Pr(Y \cup Z)$? $\frac{2}{8}$

n) Does $\Pr(Y \cup Z) = \Pr(Y) + \Pr(Z)$? yes

o) What is $\Pr(Y \cap Z)$? 0

2. An experiment consists of randomly choosing one letter from the word *mathematics*.

a) The sample space of the experiment is $S =$ $\{m, a, t, h, e, i, c, s\}$

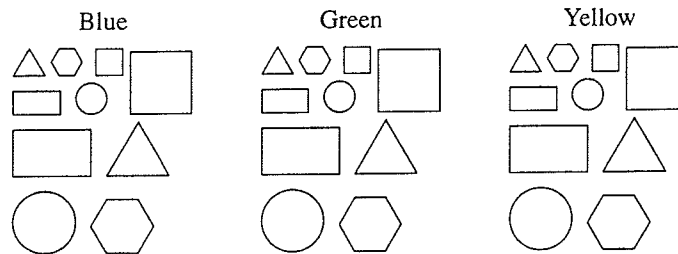
b) What is the probability of choosing h ? $\frac{1}{11}$

c) What is the probability of choosing t ? $\frac{2}{11}$

d) What is the probability of choosing either m or a ? $\frac{4}{11}$

e) Is this a uniform probability space? No - some letters have a higher probability than others.

3. Suppose an experiment consists of randomly choosing one of the pieces from the first activity:



- a) How many outcomes does this experiment have? How many simple events? 30 for both
- b) Let B be the event of choosing a blue piece.
- How many outcomes are in B ? 10
 - What is $\Pr(B)$? $\frac{1}{3}$
- c) Let L be the event of choosing a large piece.
- How many outcomes are in L ? 15
 - What is $\Pr(L)$? $\frac{1}{2}$
- d) Let M be the event of choosing a small piece.
- How many outcomes are in M ? 15
 - What is $\Pr(M)$? $\frac{1}{2}$
- e) How many outcomes are in the event $M \cap B$? 5
- f) What is $\Pr(M \cap B)$? $\frac{5}{30} = \frac{1}{6}$
- g) How many outcomes are in $M \cup B$? 20
- h) What is $\Pr(M \cup B)$? $\frac{2}{3}$
- i) What is $\Pr(M) + \Pr(B)$? $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$
- j) Is $\Pr(M \cup B) = \Pr(M) + \Pr(B)$? Why or why not?
No, because M & B overlap.

Formula: The *union rule* for probability states that if E and F are any events in a sample space S , then

$$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$$

4. Suppose that a class has 150 students and the maximum grade is 100. Eleven students' grades were 90 or more, forty-one were at least 80 but less than 90, fifty-seven were at least 60 but less than 70, and ten were less than 60. Letting x represent grades, complete the following table. (Hint: express values of x as ranges.)

Grade:	$x \geq 90$	$80 \leq x < 90$	$70 \leq x < 80$	$60 \leq x < 70$	$x < 60$
Frequency:	11	41	31	57	10
Probability:	$11/150$	$41/150$	$31/150$	$57/150$	$10/150$

a) What is the probability that a student scored at least 80?

$$\frac{41 + 11}{150} = \frac{52}{150}$$

b) What is the probability that a student scored at least 70 but less than 90?

$$\frac{31 + 41}{150} = \frac{72}{150}$$

Conditional Probability

Conditional probability refers to the probability that one event happens assuming that another event happens.

1. Students at an elementary school were surveyed to see how many siblings each one had. The following table shows the results. Complete the table by filling in the totals.

	No Siblings (E_0)	One Sibling (E_1)	Two or More Siblings (E_2)	Totals
Kindergarteners (K)	34	22	17	73
First Graders (F)	23	39	21	83
Second Graders (G)	10	25	33	68
Totals	67	86	71	224

a) How many students were surveyed? In other words, what is $n(S)$? 224

b) What is the probability that a student has no siblings? $\frac{67}{224}$

c) What is the probability that a student is in second grade? $\frac{68}{224}$

d) What is the probability that a student is in second grade and has no siblings? $\frac{10}{224}$

e) What is the probability that a second grader has no siblings? $\frac{10}{68}$

Definition: Let E and F be events. The **conditional probability** of E given F , denoted $\Pr(E|F)$, is the probability that E occurs given that F occurs. The formula is

$$\Pr(E|F) = \frac{\Pr(E \cap F)}{\Pr(F)}$$

f) What is the probability that a kindergartener has two or more siblings?

$$\frac{17}{73}$$

g) What is the probability that a student with one sibling is in first grade?

$$\frac{39}{86}$$

2. Suppose that at a party, one-third of the guests are women and two-thirds are men. Also suppose that 75% of the women wore sandals, and 20% of the men wore sandals. Consider the experiment of randomly choosing one guest at the party, and define the following events:

M = the event that the chosen guest is a man

W = the event that the chosen guest is a woman

D = the event that the chosen guest wore sandals

Determine the following probabilities.

a) $\Pr(W)$ $\frac{1}{3}$

b) $\Pr(M)$ $\frac{2}{3}$

c) $\Pr(D|W)$ $\frac{3}{4}$

d) $\Pr(D|M)$ $\frac{1}{5}$

e) What is the probability that the chosen guest is a man wearing sandals?

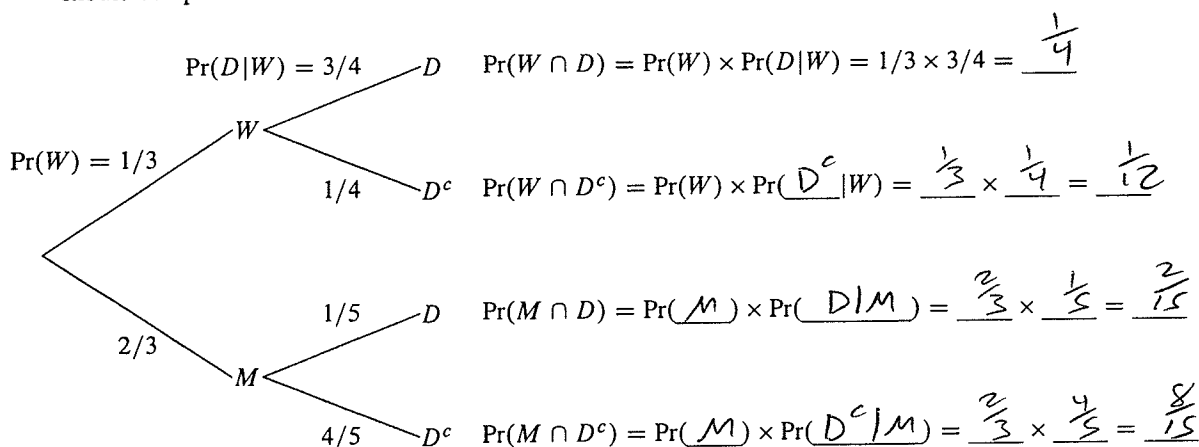
$$\frac{2}{3} \cdot \frac{1}{5} = \frac{2}{15}$$

Formula: By rewriting the formula for conditional probability, we get the **product rule**:

$$\begin{aligned} \Pr(E \cap F) &= \Pr(F) \times \Pr(E|F) \\ &= \Pr(E) \times \Pr(F|E) \end{aligned}$$

f) What is the probability that the chosen guest is a woman not wearing sandals? $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

g) We use **tree diagrams** to represent many experiments. Each branch represents a possible outcome of the experiment. Each piece of each branch between two nodes can be labeled with the probability that the event on the right side occurs given that the event on the left occurs. This is illustrated in the tree diagram below, which you should complete.



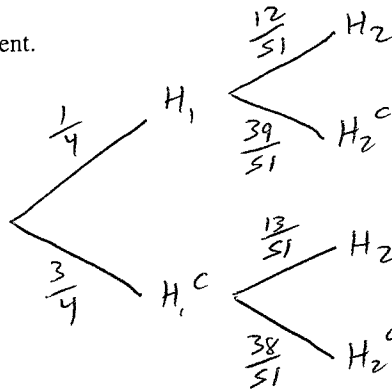
h) What is $\Pr(D)$? (Hint: What is the probability that the guest is a woman wearing sandals? What about a man wearing sandals? You now have two probabilities that can be found on the tree diagram. How should you combine them to answer the question?)

$$\frac{1}{4} + \frac{2}{15}$$

Remark: Any experiment that occurs in steps, or can be represented in steps, lends itself well to representation by a tree diagram. Each "column" in the diagram, then, will list the possible outcomes for the corresponding step in the experiment.

3. Suppose that two cards are randomly drawn in succession from a standard 52-card deck. Let H_1 be the event that the first card is a heart, and H_2 be the event that the second card is a heart.

a) Draw a tree diagram for this experiment.



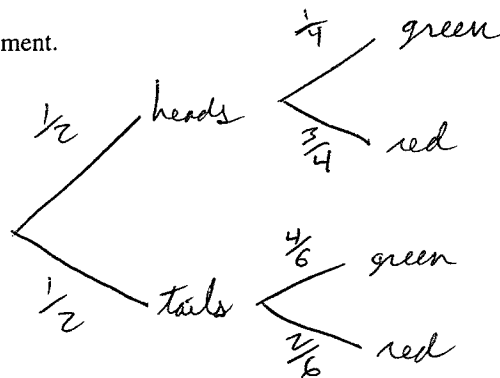
b) What is $\Pr(H_2|H_1)$? What does this probability represent (in words)?

It is $\frac{12}{51}$, the probability that the second card is a heart, given that the first one is a heart

c) What is $\Pr(H_1 \cap H_2)$? Describe the event $H_1 \cap H_2$ in words.

4. Suppose that a cup has one green marble and three red marbles, and that a bowl has four green marbles and two red marbles. An experiment consists of flipping a coin, and then randomly choosing a marble from the cup if it is heads, and from the bowl if it is tails.

a) Draw a tree diagram for this experiment.



b) Given that the coin flip is heads, what is the probability of choosing a green marble?

$$\frac{1}{4}$$

c) What is the probability of choosing a red marble?

$$\frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{2}{6}$$

5. Consider the marble experiment from problem 4. How would you find the probability that the marble came from the cup, given that it is red?

Take the probability that it is red and came from the cup and divide that by the probability that it is red.