Week 4: §5.4–§6.3

Binomial Probability

Definition: A ***binomial probability experiment*** is a probability experiment that:

- consists of n identical trials;
- has only two outcomes, called ***success*** and ***failure***; and
- has equal probability of success in each trial.

We use p to denote the probability of success, and q to denote the probability of failure. Note that $q = 1 - p$. The trials in a binomial experiment are sometimes called ***Bernoulli trials***.

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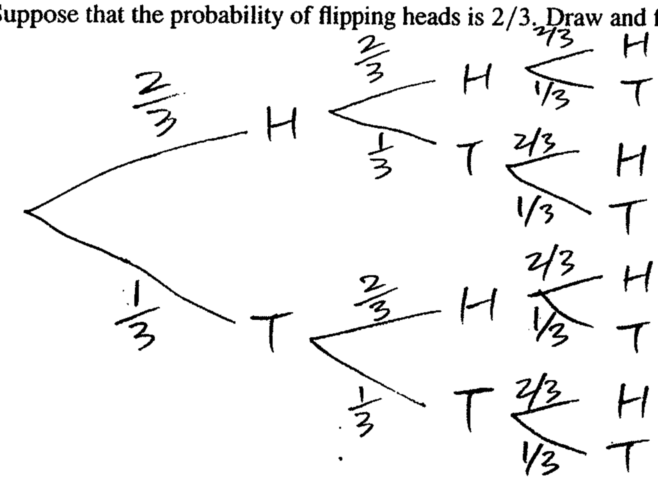
1. Suppose that a weighted coin is flipped three times.

a) Is this a binomial experiment? Why or why not?

Yes.

b) Suppose that the probability of flipping heads is $2/3$. Draw and fully label a tree diagram for this experiment.



c) How does the tree diagram reflect whether or not this is a binomial experiment?

Each opening reads $2/3$ H, $1/3$ T regardless of position, showing it is a binomial experiment.

d) How many paths lead to $\underbrace{\text{flipping no heads?}}_{1}$ What about $\underbrace{\text{exactly one?}}_{3}$ $\underbrace{\text{Exactly two?}}_{3}$

e) Express the answers to the previous problem in terms of $C(n,r)$ for some values of n and r.

$C(3,0)$, $C(3,1)$, $C(3,2)$

f) Complete the probability distribution table where x is the number of heads flipped.

x	0	1	2	3
$\Pr(x)$	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$3\left(\frac{1}{3}\right)^2\frac{2}{3} = \frac{6}{27}$	$3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{12}{27}$	$\left(\frac{2}{3}\right)^3 = \frac{8}{27}$

Formula: The probability of getting exactly k successes in a binomial experiment with n trials is

$$C(n,k)p^k q^{n-k}.$$

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2. If the coin is flipped 20 times in the experiment from the first problem, find the following probabilities.

a) The probability that exactly 3 heads are flipped.

$C(20,3)\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^{17}$

b) The probability that exactly 10 heads are flipped.

$C(20,10)\left(\frac{2}{3}\right)^{10}\left(\frac{1}{3}\right)^{10}$

c) The probability that at most 3 heads are flipped.

$C(20,3)\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^{17} + C(20,2)\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{18}$
$+ C(20,1)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{19} + \left(\frac{1}{3}\right)^{20}$

d) The probability that at least 3 heads are flipped.

$1 -$ (the answer in (c))

Random Variables

Definition: A ***random variable*** of an experiment is a rule that assigns a real number to each outcome of an experiment. We classify them using three groups, depending on the set of possible values that the variable takes:

- A ***finite discrete***, or simply ***finite***, random variable is one that can assume only finitely many values.
- An ***infinite discrete*** random variable is one that assumes infinitely many values that can be written in a sequence. For example, a random variable that takes on the values 1, 2, 3, 4, . . . is infinite discrete.
- A ***continuous*** random variable is one that can assume any value in a specified interval. For example, a random variable that can assume any value in the interval [0, 1].

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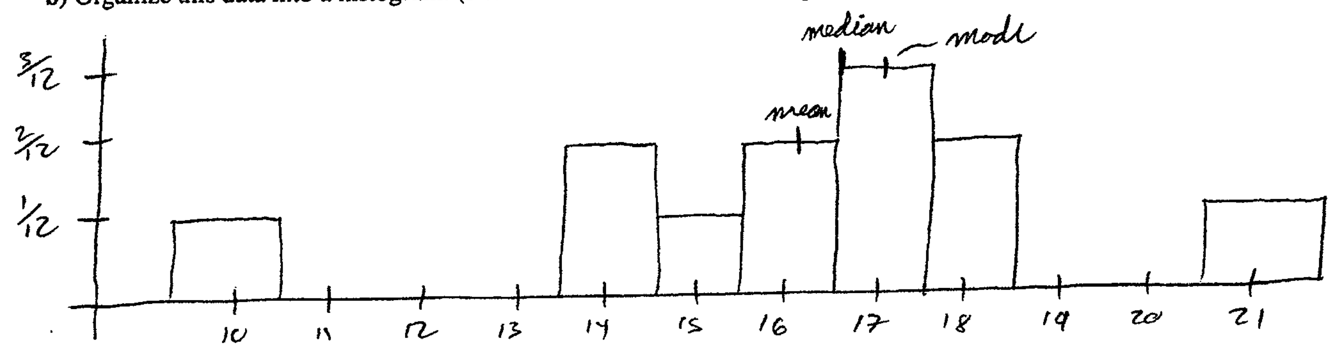
1. Find the number of letters in the name of each student in this class. Suppose that we define a random variable X to be the number of letters.

a) Is X finite, infinite discrete, or continuous? Why?

It is finite discrete, since there are only finitely many students and letters. (Suppose #s are 17, 16, 21, 17, 14, 15, 10, 18, 17, 18, 16, & 14.)

b) Organize this data into a histogram. (Remember that the values of X go on the x-axis.)



c) Find the mean number of letters, and label on the histogram.

$\frac{193}{12}$ (≈ 16.1)

Definition: The ***median*** of a collection of numbers is the number that appears in the middle of the collection when it is arranged in order of size.

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d) Find the median, and label on the histogram.

10 14 14 15 16 $\underbrace{16 \quad 17}_{16.5}$ 17 17 18 18 20

Definition: The ***mode*** of a collection of numbers is the number that appears most frequently.

e) Find the mode, and label on the histogram.

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f) Organize the data from this experiment into a frequency table.

X	10	14	15	16	17	18	21
$Pr(X=x)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

Definition: Suppose X is a random variable that takes on the values x_1 through x_n. The ***expected value*** X, denoted $E(X)$, is the sum of each value of the random variable multiplied by its relative frequency:

$$E(X) = x_1 \cdot \Pr(X = x_1) + x_2 \cdot \Pr(X = x_2) + \cdots + x_n \cdot \Pr(X = x_n),$$

where $\Pr(X = x)$, or simply $\Pr(x)$, denotes the probability that $X = x$.

g) Find the expected value of the random variable of this experiment.

$$10\left(\tfrac{1}{12}\right) + 14\left(\tfrac{2}{12}\right) + 15\left(\tfrac{1}{12}\right) + 16\left(\tfrac{2}{12}\right) + 17\left(\tfrac{3}{12}\right) + 18\left(\tfrac{2}{12}\right) + 21\left(\tfrac{1}{12}\right)$$

$$= \frac{193}{12} \approx 16.1$$

h) Is the expected value of a random variable related to one of the measures of central tendency (mean, median, and mode)?

It is equal to the mean.

Definition: Suppose X takes on the values x_1 through x_n. The ***variance*** of X, denoted $\mathrm{Var}(X)$, is the following sum:

$$\mathrm{Var}(X) = (x_1 - \mu)^2 \Pr(x_1) + (x_2 - \mu)^2 \Pr(x_2) + \cdots + (x_n - \mu)^2 \Pr(x_n),$$

where μ is the expected value of X.

i) Find the variance of the random variable of this experiment.

$(10-16.1)^2(\frac{1}{12}) + (14-16.1)^2(\frac{2}{12}) + (15-16.1)^2(\frac{1}{12}) + (16-16.1)^2(\frac{2}{12}) + (17-16.1)^2(\frac{3}{12})$

$+(18-16.1)^2(\frac{2}{12}) + (21-16.1)^2(\frac{1}{12}) \approx 7.36$

Definition: The ***standard deviation*** of a random variable X is $\sqrt{\mathrm{Var}(X)}$.

j) Find the standard deviation of the random variable of this experiment.

$\approx \sqrt{7.36} \approx 2.71$

2. Suppose an experiment consists of tossing two fair coins repeatedly. The number of heads at each step is recorded. The first three trials are recorded in the table below.

a) Complete the table by conducting the remaining seven trials.

	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6	Trial 7	Trial 8	Trial 9	Trial 10
Outcome	HH	HT	TT	HT	TH	HH	HT	TH	HH	TT
Total Heads	2	3	3	4	5	7	8	9	11	11
Total Trials	1	2	3	4	5	6	7	8	9	10
Mean	2/1	3/2	3/3	4/4	5/5	7/6	8/7	9/8	11/9	11/10

b) Does the mean appear to approach a particular value?

It appears to approach 1.

c) Consider conducting one trial of this experiment, and complete the following table. What is $E(X)$?

Outcome	$X =$ # of Heads	$\Pr(X)$
Two Heads	2	$\frac{1}{4}$
One Head, One Tail	1	$\frac{2}{4}$
Two Tails	0	$\frac{1}{4}$

$C(2,2)/2^2$ $\quad C(2,2)\,\frac{1}{2}^2\,\frac{1}{2}^0$

$C(2,1)/2^2$ $\quad C(2,1)\,\frac{1}{2}^1\,\frac{1}{2}^1$

$C(2,0)/2^2$ $\quad C(2,0)\,\frac{1}{2}^0\,\frac{1}{2}^2$

$E(X) = 2\cdot\frac{1}{4} + 1\cdot\frac{2}{4} + 0\cdot\frac{1}{4} = 1$

$S = \{HH, HT, TH, TT\}$

3. Suppose that an office has seven employees: five male and two female. Each week, a delegation of two is selected to report to their boss.

a) Complete the following table. What is the expected number of women in the delegation?

Outcome	X = # of Women	Pr(X)	
Two Women	2	$1/21$	$C(2,2)/C(7,2)$
One Woman, One Man	1	$10/21$	$C(2,1)C(5,1)/C(7,2)$
Two Men	0	$10/21$	$C(5,2)/C(7,2)$

$$E(X) = \frac{2}{21} + \frac{10}{21} + \frac{0}{21} = \frac{12}{21}$$

b) Office records show that in the past year, there were three delegations of two women, 26 of one woman and one man, and 23 of two men. Are women represented fairly on the delegations? Are men?

This yields probabilities of $\frac{3}{52}$, $\frac{26}{52}$, and $\frac{23}{52}$, which are close to the probabilities of 2, 1, or 0 women, respectively, so women and men are fairly represented.

4. Consider an experiment with outcomes and associated frequencies given in the following table. Let the random variable X be the numerical outcome. (For example, $X(0) = 0$.) Complete the table and find $E(X)$.

x	0	2	4	6	8	10	
Frequency	50	5	10	5	20	10	$\rightarrow 100$
Pr($X = x$)	$1/2$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{5}$	$\frac{1}{10}$	

$$E(X) = 0\cdot\tfrac{1}{2} + 2\cdot\tfrac{1}{20} + 4\cdot\tfrac{1}{10} + 6\cdot\tfrac{1}{20} + 8\cdot\tfrac{1}{5} + 10\cdot\tfrac{1}{10}$$

5. A ball is drawn from a basket that contains eight balls numbered 1 through 8. Let the random variable X be 3 if the number on the ball is odd, and one-half the number on the ball if the **ball's** number is even. Complete the following table, and find $E(X)$.

x	1	2	3	4
Frequency	1	1	5	1
$\Pr(X = x)$	$1/8$	$1/8$	$5/8$	$1/8$

Outcome	X
1	3
2	1
3	3
4	2
5	3
6	3
7	3
8	4

$$E(X) = 1\cdot\tfrac{1}{8} + 2\cdot\tfrac{1}{8} + 3\cdot\tfrac{5}{8} + 4\cdot\tfrac{1}{8}$$

6. Consider the collection numbers 46, 47, 48, 49, 50, 51, 52, 53, and 54.

a) What is the mean?

$(46+54) + (47+53) + (48+52) + (49+51) + 50 = 450$

$450/9 = 50$

b) What is the median?

50

c) What is the mode?

no mode

d) Which of the numbers can you increase without changing the median? Which can you decrease? Why do these changes not affect the median?

Any of the numbers larger than 50 can be increased, and any of the numbers smaller can be decreased, since 50 would remain in the middle.

Odds

Suppose that you and a friend are at a racetrack. Your friend wants to bet you \$1 that horse A will win—i.e. he will pay you \$1 if horse A loses. The odds in favor of A are 2: 5. We want to find that amount you should you agree to pay him if horse A wins to make this bet fair. (A bet is called ***fair*** if the expected winning is \$0.)

Formulas:

- If the odds in favor of an event E are $a\colon b$, then $\Pr(E) = \dfrac{a}{a+b}$.
- If the odds against an event E are $a\colon b$, then $\Pr(E^c) = \dfrac{b}{a+b}$.
- The odds in favor of an event E are $\Pr(E)\colon\Pr(E^c)$.
- The odds against an event E are $\Pr(E^c)\colon\Pr(E)$.

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1. What is the probability that horse A will win?

$2/7$

2. Suppose b is the amount (in dollars) that you agree to pay your friend if horse A wins. Complete the following table.

Outcome	$X =$ Amount You Win	$P(X)$
Horse A Wins	b	$2/7$
Horse A Loses	-1	$5/7$

3. Find your expected winnings (i.e. the expected value of X) in terms of b.

$$b \cdot \frac{2}{7} + \frac{-5}{7} = \frac{2b-5}{7}$$

4. What should you agree to pay to make the bet fair?

We need $2b-5=0$, so $b = \$2.50$.