CHEAT SHEET FOR DIFFERENTIAL EQUATIONS CHAPTERS 4 AND 6
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This handout summarizes the main computational and theoretical results from Chapters 4 and 6 of differential equations. Reading this sheet is NOT a substitute for reading the book, going to lecture, going to discussion section, or doing the homework. (This sheet may not make sense until you do at least one of those things.)

For the rest of this sheet, "I" will mean an open interval (a, b) where a could be $-\infty$ and b could be $+\infty$.

Theory

Existence and uniqueness theorem: Suppose $p_1(t), \dots, p_n(t), f(t)$ are all defined and continuous on I. Then for any single point t_0 in I and any numbers $\gamma_0, \dots, \gamma_{n-1}$, there is a solution $y(t)$ defined on I to the initial value problem:

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \cdots + p_n(t)y(t) = f(t); \qquad y(t_0) = \gamma_0, \dots, y^{(n-1)}(t_0) = \gamma_{n-1},$$

and if y_2 is a second solution defined on (at least) all of I, then $y = y_2$ on all of I.

[The $n = 2$ homogeneous case is Thm. 1 of §4.2; the $n = 2$ inhomogeneous is Thm. 4 of §4.5; arbitrary n is Thm. 1 of §6.1. If you assume $y, p_1, \dots, p_n, f$ can each be expanded as a power series centered at t_0, then you can prove this by plugging in and patiently manipulating the coefficients. Proving this in general needs crazy amounts of epsilons and deltas, which is done in (the omitted) Chapter 13.]

Wronskian theorem, pt. 1: If the functions $y_1(x), \dots, y_n(x)$ are all $n-1$-times differentiable on an interval I, and their Wronskian is nonzero at at least one point in I, then they are linearly independent (on I). Otherwise, the test is inconclusive.

[The $n = 2$ case is Lemma 1 of §4.2; arbitrary n is buried in §6.1. The proof is straight out of linear algebra; make sure you understand why.]

Wronskian theorem, pt. 2: Suppose the functions $y_1(x), \dots, y_n(x)$ are all defined on an interval I, and all solve the same homogeneous differential equation:

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \cdots + p_n(t)y(t) = 0$$

where $p_1, \dots, p_n$ are all continuous on I. Then their Wronskian is either 0 everywhere on I, or 0 nowhere on I. In the first case, $y_1, \dots, y_n$ are dependent. In the second case, $y_1, \dots, y_n$ are independent (see pt. 1).

The $n = 2$ case is Ex. 4.2.34; arbitrary n is Thm. 3 in §6.1. The proof of pt. 2 goes like this: First, letting $W(x)$ be shorthand for $W[y_1, \dots, y_n](x)$, show that $W(x)$ satisfies Abel's equation:

$$W'(x) = -p_1(x)W(x).$$

(Abel's equation for $n = 3$ case is in Ex. 6.1.30. The general case is tricky.) By integrating factors (see MATH 1B) the only solutions defined on I to the previous equation are $W(x) = Ce^{-\int p_1(x)\,dx}$ for some constant C. The previous sentence is a simplified restatement of Abel's identity (which is misstated in the text; it's missing a minus sign.) This shows that the Wronskian is either zero nowhere or zero everywhere; to get the final part you use the Existence and Uniqueness theorem.

Dimension theorem: If $p_1(t), \dots, p_n(t)$ are all continuous on the same interval I, then the collection of all solutions to:

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \cdots + p_n(t)y(t) = 0$$

defined on I is a real vector space of dimension n under usual addition and scalar multiplication.

[The $n = 2$ case is kind of Thm. 2 of §4.2; arbitrary n is kind of what you get by putting together Thms. 2 and 3 in §6.1 with Ex. 6.1.26. The proof of the "vector space" part is straight out of linear algebra (make sure you see why). Thm. 2 of §6.1 basically says "dimension is at most n" and needs Existence and Uniqueness to prove it. Thm. 3 and Ex. 6.1.26 together basically say "dimension is at least n" and also needs Existence and Uniqueness to prove them.]

Superposition principle: If $y_1(t)$ and $y_2(t)$ solve:

$$p_0(t)y_1^{(n)}(t) + p_1(t)y_2^{(n-1)}(t) + \cdots + p_n(t)y_1(t) = f_1(t), \qquad p_0(t)y_1^{(n)}(t) + p_1(t)y_2^{(n-1)}(t) + \cdots + p_n(t)y_1(t) = f_2(t)$$

then for all constants c_1 and c_2, $y(t) = c_1y_1(t) + c_2y_2(t)$ solves:

$$p_0(t)y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \cdots + p_n(t)y(t) = c_1f_1(t) + c_2f_2(t).$$

As a special case, to find the general solution to an inhomogeneous linear equation, find a particular solution to it and add it to the general solution to the homogeneous equation. [The $n = 2$ case is Thm. 3 of §4.5; arbitrary n is buried in §6.1. The proof is straight out of linear algebra; make sure you see why.]

Computations

Spring equation: The equation of motion of an object attached to a spring is:

$$my''(t) + by'(t) + ky(t) = F_{ext}(t),$$

where $y(t)$, m, b, k, $F_{ext}(t)$ are the object's position to the right of equilibrium; m is the object's mass; b is the friction coefficient; k is the spring constant; and the sum of all external forces. [See §4.1]

Solutions to const. coeff. homog. lin. diff. eqs: Suppose $a_n, \ldots, a_0$ are all constants with $a_n \neq 0$. If r is a real root of multiplicity k of the auxiliary polynomial:

$$a_n r^n + \cdots + a_2 r^2 + a_1 r + a_0$$

then e^{rt}, ..., $t^{k-1}e^{rt}$ solve the corresponding:

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0.$$

If $\alpha \pm \beta i$ are nonreal roots of multiplicity k of the auxiliary polynomial, then $e^{\alpha t}\cos\beta t$, $e^{\alpha t}\sin\beta t$, $te^{\alpha t}\cos\beta t$, $te^{\alpha t}\sin\beta t$, ..., $t^{k-1}e^{\alpha t}\cos\beta t$, $t^{k-1}e^{\alpha t}\sin\beta t$ solve the differential equation. These two rules will give n linearly independent solutions. [§4.2 does $n = 2$ with real roots; §4.3 does $n = 2$ with non-real roots; §6.2 does all n.]

Undetermined coefficients: The differential equation:

$$a_n y^{(n)}(t) + \cdots + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t),$$

where $f(t) = e^{rt}$(degree d polynomial), will have a particular solution of the form:

$$t^{\#\text{ of times } r \text{ is a root of the aux. poly.}} e^{rt}(\text{some polynomial of degree} \leq d).$$

If instead $f(t) = e^{\alpha t}(\text{deg. } d_1 \text{ polynomial})\sin\beta t + e^{\alpha t}(\text{deg. } d_2 \text{ polynomial})\sin\beta t$, then let $d = \max(d_1, d_2)$. Then the equation will have a particular solution of the form:

$$t^{\#\text{ of times } \alpha+\beta i \text{ is a root of the aux. poly}} e^{\alpha t}\Big((\text{poly. of deg.} \leq d)\cos\beta t + (\text{possibly different poly. of deg.} \leq d)\sin\beta t\Big).$$

For linear combinations of the previous two cases, use the superposition principle. [§§4.4-4.5 only treat the case $n = 2$. The formulas I wrote above work for all n. The general case is in the omitted §6.3.]

Variation of parameters: To find a particular solution to:

$$a_n y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = f(t),$$

where $a_{n-1}(t), \ldots, a_0(t)$ are all continuous on the same interval I, first find n linearly independent $y_1(t), \ldots, y_n(t)$ defined on I that solve the corresponding homogeneous equation [see the theory section for why they exist; there's no general way to find them if $a_{n-1}(t), \ldots, a_0(t)$ aren't all constant]. Solve for $v_1'(t), \ldots, v_n'(t)$ in the following equation:

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1'(t) \\ \cdots \\ v_{n-1}'(t) \\ v_n'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ f(t)/a_n \end{bmatrix},$$

then integrate; a particular solution is then $y_1(t)v_1(t) + \cdots + y_n(t)v_n(t)$.

For $n = 2$, by using Cramer's rule, this simplifies to: Let y_1 and y_2 be any linearly independent solutions to $ay'' + by' + cy = 0$. Then a particular solution to $ay'' + by' + cy = f$ is

$$-y_1 \int \frac{fy_2}{a(y_1y_2' - y_1'y_2)}\,dt + y_2 \int \frac{fy_1}{a(y_1y_2' - y_1'y_2)}\,dt.$$

[The $n = 2$; a_1 and a_0 constant case is in §4.6. The $n = 2$; a_1 and a_0 not constant case is in the omitted §4.7. (The course book's §4.7 is §4.8 in the original book.) The case of n arbitrary is in the omitted §6.4.]

Energy integral lemma: Suppose $f(x)$ is a continuous function, and $F(x)$ is any antiderivative of $f(x)$. Then any solution $y(t)$ to $y''(t) = f(y(t))$ must satisfy $\frac{1}{2}(y'(t))^2 - F(y(t)) = K$ for some constant K. [This is from the course book's §4.7, which is §4.8 in the original book.]

Misc: §4.7 also says how to get qualitative information about nonlinear systems by approximating them as linear systems. §4.8 gives vocabulary about and in-depth analysis of solutions to the homogeneous spring equation. §4.9 gives vocabulary about and in-depth analysis of solutions to the the following inhomogeneous spring equations:

$$my'' + by' + ky = F_0\cos\gamma t, \qquad my'' + by' + ky = mg,$$

where F_0 and γ are nonnegative constants and g is the (nonnegative) gravitational constant. [The course book's §§4.7-4.9 are §§4.8-4.10 in the original textbook.]