CHEAT SHEET FOR DIFFERENTIAL EQUATIONS CHAPTER 9
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This handout summarizes the main computational and theoretical results from Chapter 9 of differential equations. Reading this sheet is NOT a substitute for reading the book, going to lecture, going to discussion section, or doing the homework. (This sheet may not make sense until you do at least one of those things.)

For the rest of this sheet, "I" will mean an open interval (a, b) where a could be $-\infty$ and b could be $+\infty$.

Theory

Existence and uniqueness theorem: Suppose $\mathbf{A}(t)$ is an $n \times n$ matrix of functions and $\mathbf{f}(t)$ is an $n \times 1$ column vector of functions, all of which are defined and continuous on I. Then for any t_0 in I and $\mathbf{x}_0$ in $\mathbb{R}^n$, there is a solution $\mathbf{x}(t)$ defined on I to the initial value problem:

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t); \qquad \mathbf{x}(t_0) = \mathbf{x}_0,$$

and if $\mathbf{x}_2(t)$ is a second solution defined on (at least) all of I, then $\mathbf{x}(t) = \mathbf{x}_2(t)$ on all of I.

[This is thm. 2 of §9.4. If you assume each of the entries of $\mathbf{A}(t)$ and $\mathbf{f}(t)$ can be expanded as a power series centered at t_0, then you can prove this by plugging in and patiently manipulating the coefficients. Proving this in general needs crazy amounts of epsilons and deltas, which is done in (the omitted) Chapter 13.]

Wronskian theorem, pt. 1: Suppose that $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ are each column vector of n functions defined on I. If their Wronskian is nonzero at at least one point in I, then they are linearly independent (on I). Otherwise, the test is inconclusive.

[The proof is buried in §9.4, and is straight out of linear algebra; make sure you understand why.]

Wronskian theorem, pt. 2: Suppose that $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ are each column vectors of n functions, all defined on an interval I, and all solve the same homogeneous differential equation:

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$$

where each entry of $\mathbf{A}(t)$ is continuous on I. Then their Wronskian is either 0 everywhere on I, or 0 nowhere on I. In the first case, $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ are dependent. In the second case, $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ are independent (see pt. 1).

The proof of pt. 2 is buried in §9.4, and parts are in Exs. 9.4.32 and 9.4.33. It goes like this: First, letting $W(t)$ be shorthand for $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t)$, show that $W(t)$ satisfies Abel's equation:

$$W'(t) = \big(a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t)\big)W(t),$$

where $a_{11}(t)$ is the $(1, 1)$ entry of $\mathbf{A}(t)$, etc. (Abel's equation for $n = 3$ case is in Ex. 9.4.32. The general case is tricky.) By integrating factors (see MATH 1B) the only solutions defined on I to the previous equation are $W(y) = Ce^{\int (a_{11}(t)+a_{22}(t)+\cdots+a_{nn}(t))dt}$ for some constant C. The previous sentence is a simplified restatement of Abel's identity. This shows that the Wronskian is either zero nowhere or zero everywhere; to get the final part you use the Existence and Uniqueness theorem.

Dimension theorem: If $\mathbf{A}(t)$ is an $n \times n$ matrix of functions all defined and continuous on I, then the collection of all solutions $\mathbf{x}(t)$ to:

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$$

defined on I is a real vector space of dimension n under usual addition and scalar multiplication.

[This is basically Thm. 3 of §9.4. The "vector space" part is from linear algebra. The "dimension is at most n" needs the Existence and Uniqueness Theorem and the Wronskian Theorem pt. 2 to prove it. Ex. 9.4.34 gives "dimension is at least n" and needs Existence and Uniqueness to prove it.]

Because of the dimension theorem, for an $n \times n$ matrix $\mathbf{A}(t)$ of functions defined and continuous on I, we will call an $n \times n$ matrix $\mathbf{X}(t)$ whose columns are n linearly independent solutions to $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$ defined on I a ***fundamental matrix***. By the dimension theorem, there is always a fundamental matrix, and its columns are a basis for the solutions to $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$. By Wronskian theorem 2, any such fundamental matrix $\mathbf{X}(t)$ is invertible everywhere.

Superposition principle: If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ solve:

$$\mathbf{x}_1'(t) = \mathbf{A}(t)\mathbf{x}_1(t) + \mathbf{f}_1(t), \qquad \mathbf{x}_2'(t) = \mathbf{A}(t)\mathbf{x}_2(t) + \mathbf{f}_2(t)$$

then for all constants c_1 and c_2, $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ solves:

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + c_1\mathbf{f}_1(t) + c_2\mathbf{f}_2(t).$$

As a special case, to find the general solution to an inhomogeneous linear equation, find a particular solution to it and add it to the general solution to the homogeneous equation. [This is buried in §9.4; its proof is straight out of linear algebra.]

Constructing $\mathbf{A}(t)$: §9.1 says how to rewrite a single nth order equation and a system of higher-order equations as a first-order system $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$.

Solutions to diagonalizable const. coeff. homog. lin. diff. eqs: Suppose $\mathbf{A}$ is any constant matrix. If λ is a real eigenvalue of A with eigenvector $\mathbf{v}$, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$. If $\lambda = \alpha \pm \beta i$ is a pair of non-real eigenvalues of A with corresponding eigenvectors $\mathbf{a} \pm \mathbf{b}i$, then

$$\mathbf{x}_1(t) = e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b}, \qquad \mathbf{x}_2(t) = e^{\alpha t}\sin\beta t\mathbf{a} + e^{\alpha t}\cos\beta t\mathbf{b}$$

(i.e. $\operatorname{Re} e^{(\alpha+\beta i)t}(\mathbf{a}+\mathbf{b}i)$ and $\operatorname{Im} e^{(\alpha+\beta i)t}(\mathbf{a}+\mathbf{b}i)$) both solve $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$. Following this procedure will give a basis of solutions to $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ whenever $\mathbf{A}$ is (real or complex) diagonalizable. [From §§9.5-9.6.]

More generally, suppose λ is a real eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{v}$. Then for any solution $y(t)$ to $y^{(k)}(t) = \lambda y(t)$, the function $\mathbf{x}(t) = y(t)\mathbf{v}$ is a solution to $\mathbf{x}^{(k)}(t) = A\mathbf{x}(t)$. This gives a way to find all solutions of $\mathbf{x}^{(k)}(t) = A\mathbf{x}(t)$ whenever A is diagonalizable with real eigenvalues. [Not in book.]

Coupled mass-spring equation Suppose you have n frictionless objects of masses $m_1, \ldots, m_n$. A spring with spring coefficient k_1 connects a wall to the first object, a spring with spring coefficient k_2 connecting the first two objects, and so on until a spring with coefficient k_{n+1} connecting the last object to a second wall. Letting $x_i(t)$ be the distance to the right of equilibrium that the ith object is at time t and $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \cdots \\ x_n(t) \end{bmatrix}$. Then $\mathbf{x}(t)$ is a solution to $\mathbf{x}''(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A}$ is the matrix:

$$\begin{bmatrix} (-k_1-k_2)/m_1 & k_2/m_1 & 0 & \ldots & 0 & 0 \\ k_2/m_2 & (-k_2-k_3)/m_2 & k_3/m_2 & \ldots & 0 & 0 \\ 0 & k_3/m_3 & (-k_3-k_4)/m_3 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & (-k_{n-1}-k_n)/m_{n-1} & k_n/m_{n-1} \\ 0 & 0 & 0 & \ldots & k_n/m_n & (-k_n-k_{n+1})/m_n \end{bmatrix}$$

Then $\mathbf{A}$ will always be diagonalizable with negative eigenvalues $-\omega_1, \ldots, -\omega_n$. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the corresponding eigenvectors, then by the previous paragraph the solutions are:

$$\begin{aligned} \mathbf{x}(t) &= \left(c_1\cos\left(\sqrt{\omega_1}t\right)\mathbf{v}_1 + c_2\sin\left(\sqrt{\omega_1}t\right)\mathbf{v}_1\right) + \cdots + \left(c_{2n-1}\cos\left(\sqrt{\omega_n}t\right)\mathbf{v}_n + c_{2n}\sin\left(\sqrt{\omega_n}t\right)\mathbf{v}_n\right) \\ &= d_1\sin\left(\sqrt{\omega_1}t + \phi_1\right)\mathbf{v}_1 + \cdots + d_n\sin\left(\sqrt{\omega_n}t + \phi_n\right)\mathbf{v}_n. \end{aligned}$$

The ***normal modes*** are $d_1\sin\left(\sqrt{\omega_1}t+\phi_1\right)\mathbf{v}_1, \ldots, d_n\sin\left(\sqrt{\omega_n}t+\phi_n\right)\mathbf{v}_n$, and the corresponding ***frequencies*** are $\frac{\sqrt{\omega_1}}{2\pi}, \ldots, \frac{\sqrt{\omega_n}}{2\pi}$. [The setup is in §9.6, but the book gives a different and more complicated way to solve it.]

Undetermined coefficients: The differential equation:

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t), \qquad \mathbf{f}(t) = e^{rt}(\mathbf{c}_0 + t\mathbf{c}_1 + \cdots + t^k\mathbf{c}_k)$$

where $\mathbf{b}_0, \ldots, \mathbf{b}_k \in \mathbb{R}^n$, will have a particular solution of the form

$$\mathbf{x}(t) = e^{rt}(\mathbf{a}_0 + t\mathbf{a}_1 + \cdots + t^k\mathbf{a}_k)$$

unless the homogenous equation has a nonzero solution of this form, in which case you should try

$$\mathbf{x}(t) = e^{rt}(\mathbf{a}_0 + t\mathbf{a}_1 + \cdots + t^k\mathbf{a}_k + t^{k+1}\mathbf{a}_{k+1}),$$

unless the homogeneous...

If instead

$$\mathbf{f}(t) = e^{\alpha t}\left(\cos\beta t(\mathbf{c}_0 + t\mathbf{c}_1 + \cdots + t^k\mathbf{c}_k) + \sin\beta t(\mathbf{d}_0 + t\mathbf{d}_1 + \cdots + t^k\mathbf{d}_k)\right),$$

then you should try

$$\mathbf{x}(t) = e^{\alpha t}\left(\cos\beta t(\mathbf{a}_0 + t\mathbf{a}_1 + \cdots + t^k\mathbf{a}_k) + \sin\beta t(\mathbf{b}_0 + t\mathbf{b}_1 + \cdots + t^k\mathbf{b}_k)\right),$$

unless the homogeneous equation has a nonzero solution of this form, in which case you should try

$$\mathbf{x}(t) = e^{\alpha t}\left(\cos\beta t(\mathbf{a}_0 + t\mathbf{a}_1 + \cdots + t^k\mathbf{a}_k + t^{k+1}\mathbf{a}_{k+1}) + \sin\beta t(\mathbf{b}_0 + t\mathbf{b}_1 + \cdots + t^k\mathbf{b}_k + t^{k+1}\mathbf{b}_{k+1})\right),$$

unless the homogenous... [From §9.7.]

Variation of parameters: If you know that $\mathbf{X}(t)$ is a fundamental matrix for $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$, then a particular solution to $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$ is $\mathbf{x}(t) = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{f}(t)\,dt$. [From §9.7.]

Matrix Exponentials: For an $n \times n$ matrix $\mathbf{A}$, $e^{t\mathbf{A}} = I_n + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots$. The matrix exponential is the only fundamental matrix of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ whose value at $t = 0$ is $\mathbf{I}_n$. As such, if $\mathbf{X}(t)$ is any fundamental matrix of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, then $e^{t\mathbf{A}} = \mathbf{X}(t)\mathbf{X}(0)^{-1}$. If you know [e.g. from Cayley-Hamilton] that $(\mathbf{A} - \lambda\mathbf{I})^k = \mathbf{0}$ for some $\lambda \in \mathbb{R}$ and $k \geq 1$, then

$$e^{t\mathbf{A}} = e^{\lambda t}\left(\mathbf{I} + t(\mathbf{A} - \lambda\mathbf{I}) + \cdots + \tfrac{t^2}{2!}(\mathbf{A} - \lambda\mathbf{I})^2 + \cdots + \tfrac{t^{k-1}}{(k-1)!}(\mathbf{A} - \lambda\mathbf{I})^{k-1}\right).$$

[From §9.8.]

Solutions to const. coeff. homog. lin. diff. eqs: For an $n \times n$ matrix $\mathbf{A}$, to solve $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ perform the following steps. First, find the eigenvalues of $\mathbf{A}$ and their algebraic multiplicities. For each real eigenvalue λ with multiplicity k, compute $(\mathbf{A} - \lambda\mathbf{I})^k$, and find a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of $\operatorname{Nul}(\mathbf{A} - \lambda\mathbf{I})^k$. (These are the generalized eigenvectors of $\mathbf{A}$ corresponding to λ; there will always be k of them.) Then:

$$\mathbf{x}(t) = e^{\lambda t}\left(\mathbf{v}_i + t(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_i + \cdots + \tfrac{t^2}{2!}(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_i + \cdots + \tfrac{t^{k-1}}{(k-1)!}(\mathbf{A} - \lambda\mathbf{I})^{k-1}\mathbf{v}_i\right)$$

is a solution for each $1 \leq i \leq k$. If λ is instead nonreal, then the real and imaginary parts of the previous formula are solutions. In this way you can get n linearly independent solutions to $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$. [From §9.8.]