PARTIAL SOLUTIONS TO HOMEWORK #18 ON SECTION 4.7-4.8

JASON FERGUSON
JMF@MATH.BERKELEY.EDU

1. Ex. 4.7.3 *Try to predict the qualitative features of the solution to $y'' - 6y^2$ that satisfies the initial conditions $y(0) = -1$, $y'(0) = -1$. Compare with the computer-generated Figure 4.23. [Hint: Consider the sign of the spring stiffness.]*

Common mistakes. Many of the answers I received were very similar to the answer in the back of the book, in terms of specific choice of non-essential words and phrasing, equation choice, the steps skipped, and the order in which the answer presented its steps. I gave no credit to such answers, especially since the answer in the back of the book skips a lot of steps.

More specifically, the answer in the back of the book said that the spring opposes negative displacements and reinforces positive displacements. The book then said that this means the solution, while initially both negative and decreasing, will eventually stop decreasing, and will then increase to $+\infty$.

However, the answer in the back of the book did not explain why the solution will stop decreasing [instead of, say, having a horizontal asymptote or going to $-\infty$ with slower and slower speeds like $-\ln(t)$ does], or why afterwards the solution will go to $+\infty$ [instead of, say, having a horizontal asymptote].

A few answers also tried to use the energy integral lemma, but most did not use it correctly, and in any case the energy integral lemma doesn't give a lot of qualitative information about the solution. The energy integral lemma tells you that the solution $y(t)$ will also satisfy $\frac{1}{2}(y'(t))^2 - 2(y(t))^3 = K$ for some number K. Several answers then picked $K = 0$ because that simplifies calculations and makes the previous differential equation explicitly solvable. However, $K = 0$ is the wrong number; plugging in $t = 0$ into $K = \frac{1}{2}(y'(t))^2 - 2(y(t))^3$ gives $K = \frac{1}{2}(-1)^2 - 2(-1)^3 = \frac{5}{2}$. I don't think the solution of the initial value problem can be explicitly written in terms of polynomials, trig functions, exponents, and logarithms. □

Solution. We rewrite the equation as:

$$1y''(t) + 0y'(t) + (-6y(t))y(t) = 0.$$

That means that we can consider the equation as describing an undamped mass-spring system with mass $m = 1$ and nonconstant stiffness $k = -6y(t)$.

The solution has the following features: (1) At $t = 0$ it is negative and decreasing. (2) It eventually stops decreasing. (3) Then it increases, and continues to increase monotonically until it becomes positive, and then it increases to $+\infty$ as $t \to \infty$. I will explain why it does each of these three things:

The problem gives that $y(0) = -1$ and $y'(0) = -1$. That means that the solution is originally negative and decreasing. At $t = 0$, $k = -6y(0) = 6$, which is a positive stiffness. From what we know about undamped mass-spring systems, even if the stiffness stayed constant at 6, the object would eventually stop, turn around, and then return to $y = -1$. However, since the stiffness is equal to $-6y(t)$, the stiffness will be greater than 6 whenever $y(t) < -1$. So this just means that the object will stop and then return to $y = -1$ even faster than it would have if the stiffness had remained constant at $k = 6$.

I can't use the argument of the previous paragraph once $-1 \le y < 0$, since for those y, we will have $0 < k \le 6$. At any given point t, the spring will have positive stiffness and so will pull the object toward the origin. However, as the object is pulled to the origin the stiffness will decrease, so it could hypothetically be the case that the stiffness never catches up and never is strong enough to bring the object all the way to the origin.

But the previous hypothetical situation doesn't happen: After the object returns to $y = -1$, it will have some positive velocity. That means even if the object's speed stayed constant, it would eventually reach the x-axis. But whenever $y(t) < 0$, the stiffness $k = -6y(t)$ is positive, which means that the spring is pulling the object towards $y(0)$. That means the object's speed isn't constant; instead it's increasing. So the object will reach $y = 0$ even faster than if its velocity had just stayed constant.

Once the object crosses the t-axis, i.e. y becomes positive, $k = -6y$ will be negative. This can't happen for real-world springs, but it physically means that the "spring" pushes the object away from the center. That means that once y becomes positive, the spring will continue to push the object away from $y = 0$, i.e. make it more positive. In fact, since the spring is applying a positive force to the object, the object will accelerate, i.e. its speed will increase, so it will go to $+\infty$ as $t \to +\infty$ at an even faster rate than if its speed had just stayed constant at what it was when it first crossed the t-axis. □

2. Ex. 4.7.12. *Use reduction of order to derive the solution $y_2(t)$ in equation* (5) *for Legendre's equation.*

The equation:

$$(1-t^2)y'' - 2ty' + 2y = 0 \tag{2}$$

also has the superposition property. It is a linear variable-coefficient equation and is a special case of Legendre's equation $(1-t^2)y'' - 2ty' + \lambda y = 0$...

...

The Legendre equation (2) has one simple solution $y_1(t) = t$, as can easily be verified by mental calculation. A second, linearly independent, solution for $-1 < t < 1$ can be derived. Traditionally, the second solution is taken to be:

$$y_2(t) = \frac{t}{2}\ln\left(\frac{1+t}{1-t}\right) - 1. \tag{5}$$

Common mistakes. Many answers tried to solve $(1-t^2)y'' - 2ty' + \lambda y = 0$ instead of $(1-t^2)y'' - 2ty' + 2y = 0$. The text explicitly refers you to $(1-t^2)y'' - 2ty' + 2y = 0$ [I recopied that part of the text above] and in fact the only number λ for which $y_1(t) = t$ solves $(1-t^2)y'' - 2ty' + \lambda y = 0$ is $\lambda = 2$.

A lot of answers stopped at $(1-t^2)tv''(t) + (2-4t^2)v'(t) = 0$, presumably because you didn't know or remember how integrating factors worked. Many answers didn't show their work in the integrations, or turned $|t-1|$ into $t-1$ instead of $1-t$ [the problem said $-1 < t < 1$].

Finally, in the computations you need to divide by t. Since t could be 0 I wanted you to mention something about it. You didn't necessarily need to explain why the solution $y_2(t) = \frac{t}{2}\ln\left(\frac{1+t}{1-t}\right) - 1$ still works even at $t = 0$, but I wanted you to say something that showed you noticed that you divided by t. □

Solution 1. First, if you know one nonzero solution $y_1(t)$ to a homogeneous linear differential equation:

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0,$$

then reduction of order is a way to find a second solution $y_2(t)$ to the differential equation that is not a constant multiple of $y_1(t)$.

The way the method works is, write $y_2(t)$ as $v(t)y_1(t)$ for some unknown function $v(t)$. Plug this $y_2(t)$ into:

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0,$$

and you will eventually get a first-order equation in $v'(t)$. Solve for $v'(t)$, for example by using integrating factors, then integrate to find $v(t)$. Then the second solution $y_2(t)$ will be $v(t)y_1(t)$.

In this specific problem, we are given that $y_1(t) = t$ is a solution to:

$$(1-t^2)y'' - 2ty' + 2y = 0$$

and we want a second solution $y_2(t)$ defined on $-1 < t < 1$. So we let $y_2(t) = y_1(t)v(t) = tv(t)$. Then we have:

$$y_2(t) = tv(t), \qquad y_2'(t) = tv'(t) + v(t), \qquad y_2''(t) = tv''(t) + 2v'(t),$$

so that:

$$0 = (1-t^2)y_2'' - 2ty_2' + 2y_2 = \big((1-t^2)tv''(t) + 2(1-t^2)v'(t)\big) - \big(2t^2v'(t) + 2tv(t)\big) + \big(2tv(t)\big) = (1-t^2)tv''(t) + (2-4t^2)v'(t).$$

Therefore, we need to solve

$$(1-t^2)tv''(t) + (2-4t^2)v'(t) = 0 \tag{a}$$

for $v(t)$. We divide (a) through by $t(1-t^2)$ to get the equation:

$$v''(t) + \frac{2-4t^2}{t(1-t^2)}v'(t) = 0. \tag{b}$$

(Since $-1 < t < 1$ we can divide by $1-t^2$ with no problems. At the end of this solution I will explain what to do about $t = 0$. In the meantime, assume that t is in $(-1, 0) \cup (0, 1)$.) This is a first-order equation in $v'(t)$, so we can solve it with integrating factors.

As a reminder, integrating factors are a way to solve the first-order inhomogeneous variable-coefficient equation

$$y'(t) + a(t)y(t) = b(t)$$

for $y(t)$, for any two functions $a(t)$ and $b(t)$. [In this specific problem, $y(t) = v'(t)$, $a(t) = \frac{2-4t^2}{t(1-t^2)}$, and $b(t) = 0$.] The way integrating factors works is, you multiply through by $e^{\int a(t)\,dt}$ to get:

$$y'(t)e^{\int a(t)\,dt} + a(t)y(t)e^{\int a(t)\,dt} = b(t)e^{\int a(t)\,dt}.$$

The left-hand side is $\frac{d}{dt}\left(y(t)e^{\int a(t)\,dt}\right)$, so can be easily integrated.

Returning to the specific problem, we need to find $e^{\int \frac{2-4t^2}{t(1-t^2)}\,dt}$ and then multiply (b) through by it: That means we first need to find $\int \frac{2-4t^2}{t(1-t^2)}\,dt$, which we can do by partial fractions. We know that there are some numbers A, B, C for which:

$$\frac{2-4t^2}{t(1-t^2)} = \frac{A}{t} + \frac{B}{1-t} + \frac{C}{1+t}$$

for all $t \neq 0, \pm 1$. Multiplying through by $t(1-t^2)$ gives that:

$$2 - 4t^2 = A(1-t^2) + Bt(1+t) + Ct(1-t)$$

holds for all $t \neq 0, \pm 1$. But since these are polynomials, this means that it also holds for $t = 0$, 1, and -1, and plugging in gives:

$$2 = A, \qquad -2 = 2B, \qquad -2 = -2C,$$

so $A = 2$, $B = -1$, and $C = 1$. Therefore,

$$\int \frac{2-4t^2}{t(1-t^2)}\,dt = \int \left(\frac{2}{t} - \frac{1}{1-t} + \frac{1}{1+t}\right) dt = 2\ln|t| + \ln|1-t| + \ln|1+t| = \ln|t^2(1-t^2)| = \ln t^2(1-t^2),$$

which means the integrating factor is $t^2(1-t^2) = t^2 - t^4$. Multiplying (b) through by it gives:

(c) $$(t^2 - t^4)v''(t) + (2t - 4t^3)v'(t) = 0.$$

The left-hand side of (c) is $\frac{d}{dt}\big((t^2 - t^4)v'(t)\big)$, so integrating (c) gives:

$$(t^2 - t^4)v'(t) = a$$

for some number a. That means $v(t) = a\int \frac{dt}{t^2 - t^4}$, so we only need to find this last integral.

We use partial fractions again. This time:

$$\frac{1}{t^2 - t^4} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{1-t} + \frac{D}{1+t}$$

for some numbers A-D. Multiplying through by $t^2(1-t^2)$ gives:

$$1 = At(1-t^2) + B(1-t^2) + Ct^2(1+t) + Dt^2(1-t).$$

Then plugging in $t = 0$, $t = 1$, and $t = -1$ gives:

$$1 = B, \qquad 1 = 2C, \qquad 1 = 2D.$$

so $B = 1$ and $C = D = \frac{1}{2}$. Then plugging in $t = 2$ gives:

$$1 = -6A + 1(-3) + \tfrac{1}{2}12 - 4\tfrac{1}{2} = -6A + 1,$$

so $A = 0$. This means:

$$v(t) = a\int \frac{dt}{t^2 - t^4} = \int \left(\frac{1}{t^2} + \frac{1/2}{1-t} + \frac{1/2}{1+t}\right) dt = a\left(-\frac{1}{t} - \frac{1}{2}\ln|1-t| + \frac{1}{2}\ln|1+t|\right) + b = a\left(\frac{1}{2}\ln\frac{1+t}{1-t} - \frac{1}{t}\right) + b,$$

for some constant b, so that

$$y_2(t) = tv(t) = a\left(\frac{t}{2}\ln\frac{1+t}{1-t} - 1\right) + bt.$$

Since we want a solution that is not a constant multiple of $y_1(t) = t$, we pick $a = 1$ and $b = 0$ for simplicity, and we get that:

$$y_2(t) = \frac{t}{2}\ln\frac{1+t}{1-t} - 1,$$

is a solution to Legendre's equation on $(-1, 0) \cup (0, 1)$.

According to the problem statement, it's unclear if we need to also check that $y_2(t)$ works at $t = 0$. But for the sake of completeness, I'll say how to check it. The most straightforward way to check it is to plug $y_2(t)$ back into the original equation and check directly that it works. But another way to do it is to divide Legendre's equation through by $1 - t^2$ to get:

(d) $$y'' - \frac{2t}{1-t^2}y' + \frac{2}{1-t^2}y = 0.$$

By the existence and uniqueness theorem applied to (d) on the interval $(-1, 1)$, there is only one function $y(t)$ defined on $(-1, 1)$ that solves (d) everywhere on $(-1, 1)$ and for which $y(1/2) = y_2(1/2)$ and $y'(1/2) = y_2'(1/2)$. Similarly, there is only one function $z(t)$ defined on $(-1, 1)$ that solves (d) everywhere on $(-1, 1)$ and for which $z(-1/2) = y_2(-1/2)$ and $z'(-1/2) = z_2'(-1/2)$.

If we could show that $y(t) = z(t) = y_2(t)$, we would be done, since then $y_2(t) = y(t)$ and $y(t)$ was something that by assumption solves (d) everywhere on $(-1, 1)$. So we will focus on showing $y(t) = z(t) = y_2(t)$.

First, since $y(t)$ and $y_2(t)$ are defined and solve (d) on at least $(0, 1)$ and satisfy the same initial conditions at $1/2$ [by assumption], by the existence and uniqueness theorem applied to the interval $(0, 1)$, we know that $y(t) = y_2(t)$

everywhere on $(0,1)$. Since $y(t)$ is twice differentiable on $(-1,1)$, both $y(t)$ and $y'(t)$ are continuous at $t=0$ like $y_2(t)$ and $y_2'(t)$ are, so $y(0)=y_2(0)$ and $y'(0)=y_2'(0)$.

Similarly, by the existence and uniqueness theorem applied to the interval $(-1,0)$, we know that $z(t)=y_2(t)$ everywhere on $(-1,0)$. Since $z(t)$ is continuously differentiable on $(-1,1)$ it must be continuous at $t=0$ like $y_2(t)$ is, so $z(0)=y_2(0)$ and $z'(0)=y_2'(0)$.

This means that $y(0)=y_2(0)=z(0)$ and $y'(0)=y_2'(0)=z'(0)$. Since we know $y(t)$ and $z(t)$ both solve (d) and satisfy the same initial data, by the existence and uniqueness theorem on the interval $(-1,1)$, we get that $y(t)=z(t)$ everywhere on $(-1,1)$. Since $y_2(t)=y(t)$ on $[0,1)$ and $y_2(t)=z(t)$ on $(-1,0]$, we have that $y_2(t)=y(t)=z(t)$ everywhere on $(-1,1)$, so $y_2(t)$ solves Legendre's equation on $(-1,1)$. □

Solution 2. Prof. Grunbaum, in lecture, derived the following general formula for reduction of order: If $y_1(t)$ solves $y''(t)+p(t)y'(t)+q(t)y(t)=0$, then so does

$$y_2(t)=y_1(t)\int \tfrac{1}{(y_1(t))^2}e^{-\int p(t)\,dt}\,dt.$$

In this problem, if you divide the Legendre equation through by $1-t^2$ you get:

$$y''-\frac{2t}{1-t^2}y'+\frac{2}{1-t^2}y=0.$$

So in this problem, $y_1(t)=t$ and $p(t)=-\frac{2t}{1-t^2}$. Then by substituting $u=1-t^2$, $\int p(t)\,dt=\ln|1-t^2|+c_1=\ln(1-t^2)+c_1$, so that:

$$y_2(t)=t\int \tfrac{1}{t^2}e^{-\ln(1-t^2)+c_1}\,dt=t\int \tfrac{e^{c_1}}{t^2(1-t^2)}\,dt.$$

Then you integrate $\int \frac{dt}{t^2(1-t^2)}$ and worry about dividing by t as in Solution 1.

For Legendre's equation, the general formula is faster than doing the integrating factors, but this is not always the case. □

3. Ex. 4.8.1 *A 2-kg mass is attached to a spring with stiffness $k=50$ N/m. The mass is displaced $1/4$ m to the left of the equilibrium point and given a velocity of 1 m/sec to the left. Neglecting damping, find the equation of motion of the mass along with the amplitude, period, and frequency. How long after release does the mass pass through the equilibrium position?*

Common mistakes. Very few answers explicitly said that the problem wanted the *first* time after $t=0$ that $y(t)=0$. Even fewer explicitly said that $\frac{1}{5}\left(\pi-\arctan\frac{5}{4}\right)$ is the *first* time after $t=0$ for which $y(t)=0$, and even fewer than that correctly showed it. □

Solution. Let $y(t)$ be the rightward displacement in meters at t seconds. Then we are to solve the initial value problem $2y''+50y=0$, $y(0)=-1/4$, $y'(0)=-1$. The auxiliary polynomial is $2x^2+50$, so has roots $\pm 5i$. The general homogeneous solution is

$$y(t)=c_1\sin 5t+c_2\cos 5t,\ \text{ so } y'(t)=5c_1\cos 5t-5c_2\sin 5t.$$

Plugging in $t=0$ gives:

$$-\tfrac{1}{4}=y(0)=c_2,\qquad -1=y'(0)=5c_1,$$

so $c_1=-\frac{1}{5}$ and $c_2=-\frac{1}{4}$, which means:

$$\boxed{\text{if } y(t) \text{ is the rightward displacement in meters at } t \text{ seconds, then } y(t)=-\tfrac{1}{5}\sin 5t-\tfrac{1}{4}\cos 5t}.$$

Therefore, the period is $\boxed{\frac{2\pi}{5}\text{ sec}}$ and the frequency is $\boxed{\frac{5}{2\pi}\text{ Hz}}$.

It is possible to write such a solution as $A\sin(5t+\phi)$ for some positive number A and some number ϕ. In fact, A is the amplitude in meters, and (A,ϕ) is just the polar coordinates of $\left(-\frac{1}{5},-\frac{1}{4}\right)$, so the amplitude is

$$\sqrt{\left(-\tfrac{1}{5}\right)^2+\left(-\tfrac{1}{4}\right)^2}=\sqrt{\frac{16+25}{16\cdot 25}}=\boxed{\frac{\sqrt{41}}{10}\text{ m}}.$$

Next, $(\cos\phi,\sin\phi)=\frac{10}{\sqrt{41}}\left(-\frac{1}{5},-\frac{1}{4}\right)$. This means $\tan\phi=\frac{-1/4}{-1/5}=\frac{5}{4}$, and since ϕ is in quadrant III, $\phi=\pi+\arctan\frac{5}{4}$. Thus:

$$y(t)=\frac{\sqrt{41}}{10}\sin\left(5t+\pi+\arctan\frac{5}{4}\right)=-\frac{\sqrt{41}}{10}\sin\left(5t+\arctan\frac{5}{4}\right).$$

Therefore, $y(t)=0$ only when $5t+\arctan\frac{5}{4}=n\pi$ for some integer n, i.e.

$$t=\frac{1}{5}\left(n\pi-\arctan\frac{5}{4}\right)$$

for some integer n. This is an increasing function of n, so we want the smallest integer n that makes $\frac{1}{5}\left(n\pi - \arctan\frac{5}{4}\right)$ positive. Since $\frac{5}{4} > 0$, $0 < \arctan\frac{5}{4} < \frac{\pi}{4}$, so this smallest integer n is 1, so the first time t that the mass passes through equilibrium, i.e. the smallest positive t with $y(t) = 0$, is $\boxed{\frac{1}{5}\left(\pi - \arctan\frac{5}{4}\right) \text{ sec}}$. □

4. Ex. 4.8.9: *A 2-kg mass is attached to a spring with stiffness 40 N/m. The damping constant for the system is $8\sqrt{5}$ N-sec/m. If the mass is pulled 10 cm to the right of equilibrium and given an initial rightward velocity of 2 m/sec, what is the maximum displacement from equilibrium that it will attain?*

Common mistakes. Very few answers noticed that the question asked about maximum displacement from equilibrium, i.e. $|y(t)|$, instead of maximum righward displacement, i.e. $y(t)$. Since $y(t)$ ends up being positive for all $t \geq 0$ the answers are the same, but you still need to say something about it.

Most answers correctly found the critical point, but only a few answers explicitly said that this critical point is a maximum, and even fewer correctly showed it. □

Solution. It will turn out that this system is critically damped. In this solution, for ease of writing I will find the maximum displacement from equilibrium for a critically damped system whose auxiliary polynomial has double root $-r$ where $r > 0$, has $y(0) = A > 0$, and has $y'(0) = B > 0$ in terms of r, A, and B. Then I will substitute $r = 2\sqrt{5}$, $A = \frac{1}{10}$, and $B = 2$ to get the final answer.

Let $y(t)$ denote the number of meters to the right of equilibrium that the mass is after t seconds. Then we are to find the maximum value of $|y(t)|$, where $y(t)$ is the solution to the initial value problem

$$my''(t) + by'(t) + ky(t) = 0$$

with:

$$m = 2, \qquad b = 8\sqrt{5}, \qquad k = 40, \qquad y(0) = A, \qquad y'(0) = B,$$

where $A = \frac{1}{10}$ and $B = 2$. $A = \frac{1}{10}$ because the initial position was 10 centimeters, which is 0.1 meters.

We first solve the differential equation. The characteristic polynomial is:

$$2x^2 + 8\sqrt{5}x + 40 = 0,$$

and its roots are:

$$\frac{-8\sqrt{5} \pm \sqrt{\left(8\sqrt{5}\right)^2 - 4 \cdot 2 \cdot 40}}{2 \cdot 2} = \frac{-8\sqrt{5} \pm \sqrt{320 - 320}}{4} = -2\sqrt{5}.$$

Therefore, the characteristic polynomial has $-r = -2\sqrt{5}$ as a repeated root, so the solutions to the differential equation $my''(t) + by'(t) + ky(t) = 0$ are:

$$y(t) = c_1 e^{-rt} + c_2 t e^{-rt} = (c_1 + c_2 t)e^{-rt}.$$

for any real numbers c_1 and c_2.

Now we use the initial conditions to solve for c_1 and c_2. First,

$$y'(t) = -rc_1 e^{-rt} + c_2 e^{-rt} - c_2 r t e^{-rt} = e^{-rt}(c_2 - rc_1 - c_2 rt),$$

so

$$y(0) = c_1 \cdot 1 + c_2 \cdot 0 \cdot 1 = c_1, \qquad y'(0) = 1(c_2 - rc_1 - c_2 r \cdot 0) \cdot 1 = c_2 - rc_1.$$

Therefore, $c_1 = A$ and $c_2 = B + rc_1 = B + Ar$.

Because A, B, and r are positive, c_1 and c_2 are both positive. Therefore, $y(t) = (c_1 + c_2 t)e^{-rt} \geq 0$ for all $t \geq 0$, so to find the largest displacement it suffices to maximize $y(t)$ for $t \geq 0$.

From the physical description of the problem, it would seem that $y(t)$ has only one critical point for $t \geq 0$ and the critical point is the maximum value of $y(t)$. We will show this mathematically using the first derivative test.

Now,

$$y'(t) = e^{-rt}\big(-rA + (B + Ar) - (B + Ar)rt\big) = e^{-rt}\big(B - (Ar + B)rt\big).$$

Because B and $(Ar + B)r$ are positive, $y'(t) > 0$ if $t < \frac{B}{(Ar+B)r}$, $y'(t) = 0$ at $t = \frac{B}{(Ar+B)r}$, and $y'(t) < 0$ for $t > \frac{B}{(Ar+B)r}$. By the first derivative test, $y(t)$ has its global maximum on $[0, \infty)$ at:

$$t = \tfrac{B}{(Ar+B)r},$$

and the value of $y(t)$ at $t = \frac{B}{(Ar+B)r}$ is:

$$(c_1 + c_2 t)e^{rt} = \left(A + (Ar + B)\tfrac{B}{(Ar+B)r}\right)\exp\left(r\tfrac{B}{(Ar+B)r}\right) = \left(A + \tfrac{B}{r}\right)\exp\left(-\tfrac{B}{Ar+B}\right).$$

Substituting $r = 2\sqrt{5}$, $A = \frac{1}{10}$, and $B = 2$ gives:

$$A + \tfrac{B}{r} = \tfrac{1}{10} + \tfrac{2}{2\sqrt{5}} = \tfrac{1+2\sqrt{5}}{10},$$
$$-\tfrac{B}{Ar+B} = -\tfrac{2}{\frac{1}{10}(2\sqrt{5})+2} = -\tfrac{20}{2\sqrt{5}+20} = -\tfrac{10}{10+\sqrt{5}} = -\tfrac{10(10-\sqrt{5})}{(10+\sqrt{5})(10-\sqrt{5})} = -\tfrac{10(10-\sqrt{5})}{95} = \tfrac{2\sqrt{5}-20}{19}.$$

The final answer is therefore:

$$\boxed{\tfrac{1+2\sqrt{5}}{10}\exp\left(\tfrac{2\sqrt{5}-20}{19}\right) \text{ meters}} \approx \boxed{0.24167 \text{ m}}. \qquad \square$$

5. Ex. 4.8.11: *A 1-kg mass is attached to a spring with stiffness 100 N/m. The damping constant for the system is 0.2 N-sec/m. If the mass is pushed rightward from the equilibrium position with a velocity of 1 m/sec, when will it attain its maximum displacement to the right?*

Common mistakes. Most answers found the correct critical point of $\frac{10}{\sqrt{9999}} \arctan\sqrt{9999}$. However, the function $y(t)$ has infinitely many critical points; very few answers explicitly said this. Also, very few answers explained why $\frac{10}{\sqrt{9999}} \arctan\sqrt{9999}$ is a local maximum, let alone why it is the absolute maximum over all $t \geq 0$.

Some answers correctly said that "the smallest nonnegative critical point will be the maximum," but very few answers explained why. Also, very few answers explicitly said that $\frac{10}{\sqrt{9999}} \arctan\sqrt{9999}$ is the smallest positive critical point, and even fewer correctly showed it. $\square$

Solution. It will turn out that this system is underdamped. In this solution, for ease of writing I will find the nonnegative time t that maximizes $y(t)$ where $y(t)$ is the solution to any underdamped system whose auxiliary polynomial has complex roots $-\alpha \pm \beta i$ where $\alpha, \beta > 0$, with $y(0) = 0$ and $y'(0) = 1$ in terms of α and β. Then I will substitute $\alpha = \frac{1}{10}$ and $\beta = \frac{\sqrt{9999}}{10}$ to get the final answer.

Let $y(t)$ denote the number of meters to the right of equilibrium that the mass is after t seconds. Then we are to find the nonnegative time t that maximizes $y(t)$, where $y(t)$ is the solution to the initial value problem

$$my''(t) + by'(t) + ky(t) = 0$$

with:

$$m = 1, \qquad b = \frac{1}{5}, \qquad k = 100, \qquad y(0) = 0, \qquad y'(0) = 1.$$

We first solve the differential equation. The characteristic polynomial is:

$$x^2 + \tfrac{1}{5}x + 100 = 0,$$

and its roots are:

$$\frac{-\frac{1}{5} \pm \sqrt{\left(\frac{1}{5}\right)^2 - 4 \cdot 1 \cdot 100}}{2 \cdot 1} = -\frac{1}{10} \pm \frac{1}{10}\sqrt{1 - 4 \cdot 25 \cdot 100} = -\frac{1}{10} \pm \frac{\sqrt{9999}}{10}i.$$

Therefore, the characteristic polynomial has $-\alpha \pm \beta i$ as nonreal roots where $\alpha = \frac{1}{10}$ and $\beta = \frac{\sqrt{9999}}{10}$, so the solutions to the differential equation $my''(t) + by'(t) + ky(t) = 0$ are:

$$y(t) = e^{-\alpha t}\big(c_1 \sin\beta t + c_2 \cos\beta t\big).$$

for any real numbers c_1 and c_2.

Now we use the initial conditions to solve for c_1 and c_2. First,

$$0 = y(0) = c_2,$$

so $y(t) = c_1 e^{-\alpha t}\sin\beta t$. Then:

$$y'(t) = -\alpha c_1 e^{-\alpha t}\sin\beta t + \beta c_1 e^{-\alpha t}\cos\beta t,$$

so $1 = y'(0) = \beta c_1$, which means $c_1 = \frac{1}{\beta}$. Then:

$$y'(t) = e^{-\alpha t}\big(-\tfrac{\alpha}{\beta}\sin\beta t + \cos\beta t\big).$$

Now, from the physical description of the problem and the graphs of the sample solutions to underdamped systems in the book, it would seem that the maximum rightward displacement would occur at the first critical point. We will show this mathematically using the first derivative test, since the book never gives a complete argument of this form.

In order to use the first derivative test, we need to find when $y'(t)$ is positive and negative. To do this, it will help to write $y'(t)$ as

$$Ae^{-\alpha t}\sin(\beta t + \phi)$$

for some real A and ϕ. In fact, we can choose (A, ϕ) as the polar coordinates of $\big(-\frac{\alpha}{\beta}, 1\big)$. Thus:

$$A = \sqrt{\left(-\tfrac{\alpha}{\beta}\right)^2 + 1^2} = \frac{\sqrt{\alpha^2 + \beta^2}}{\beta}.$$

Then $(\cos\phi, \sin\phi) = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\left(-\frac{\alpha}{\beta}, 1\right)$, so $\tan\phi = -\frac{\beta}{\alpha}$. Since ϕ is in quadrant II, this means $\phi = \pi + \arctan\left(-\frac{\beta}{\alpha}\right) = \pi - \arctan\frac{\beta}{\alpha}$. Therefore,

$$y'(t) = \frac{\sqrt{\alpha^2+\beta^2}}{\beta}e^{-\alpha t}\sin\left(\beta t + \pi - \arctan\frac{\beta}{\alpha}\right) = -\frac{\sqrt{\alpha^2+\beta^2}}{\beta}e^{-\alpha t}\sin\left(\beta t - \arctan\frac{\beta}{\alpha}\right).$$

We know that $\sin x > 0$ only when $2n\pi < x < (2n+1)\pi$ for some integer n, and $\sin x < 0$ only when $(2n-1)\pi < x < 2n\pi$ for some integer n. This means $y'(t) > 0$ only when:

$$(2n-1)\pi < \beta t - \arctan\frac{\beta}{\alpha} < 2n\pi,$$

i.e.

$$\frac{2n-1}{\beta}\pi + \frac{1}{\beta}\arctan\frac{\beta}{\alpha} < t < \frac{2n}{\beta}\pi + \frac{1}{\beta}\arctan\frac{\beta}{\alpha}$$

for some integer n. In the same way, $y'(t) < 0$ only when:

$$\frac{2n}{\beta}\pi + \frac{1}{\beta}\arctan\frac{\beta}{\alpha} < t < \frac{2n+1}{\beta}\pi + \frac{1}{\beta}\arctan\frac{\beta}{\alpha}$$

for some integer n. Then by the first derivative test, the local maxima of $y(t)$ are at the times t_n, where:

$$t_n = \frac{2n}{\beta}\pi + \frac{1}{\beta}\arctan\frac{\beta}{\alpha}$$

for each integer n.

Next, $t_n \geq 0$ only when $n \geq -\frac{1}{2\pi}\arctan\frac{\beta}{\alpha}$. But since $\frac{\beta}{\alpha} > 0$, $0 < \arctan\frac{\beta}{\alpha} < \frac{\pi}{2}$, which means

$$-\frac{1}{4} < -\frac{1}{2\pi}\arctan\frac{\beta}{\alpha} < 0.$$

Therefore, $t_n \geq 0$ only when $n \geq 0$.

Finally, substituting in $t = t_n$ into $y(t) = \frac{1}{\beta}e^{-\alpha t}\sin\beta t$ gives:

$$\begin{aligned} y(t_n) &= \frac{1}{\beta}\exp\left(-\frac{2n\alpha}{\beta}\pi + \frac{\alpha}{\beta}\arctan\frac{\beta}{\alpha}\right)\sin\left(2n\pi + \arctan\frac{\beta}{\alpha}\right) \\ &= \frac{1}{\beta}\sin\left(\arctan\frac{\beta}{\alpha}\right)\exp\left(-\frac{\alpha}{\beta}\arctan\frac{\beta}{\alpha}\right)e^{-2\alpha\pi n/\beta} \\ &= \frac{1}{\sqrt{\alpha^2+\beta^2}}\exp\left(-\frac{\alpha}{\beta}\arctan\frac{\beta}{\alpha}\right)e^{-2\alpha\pi n/\beta}. \end{aligned}$$

Since $\frac{1}{\sqrt{\alpha^2+\beta^2}}\exp\left(-\frac{\alpha}{\beta}\arctan\frac{\beta}{\alpha}\right)$ and $\frac{2\alpha\pi}{\beta}$ are positive constants, $y(t_n)$ is a decreasing function of n. This means that the global maximum of $y(t)$ on $[0,\infty)$ occurs at t_0, i.e. at:

$$t = \frac{1}{\beta}\arctan\frac{\beta}{\alpha} = \boxed{\left(\frac{10}{\sqrt{9999}}\arctan\sqrt{9999}\right)\text{ sec}} \approx \boxed{.156087\text{ sec}}.$$ ◻

Notice that the maximum value of $y(t)$ is not at one of the places where $y(t) = \frac{1}{\beta}e^{-\alpha t}\sin\beta t$ touches the graph of $\frac{1}{\beta}e^{-\alpha t}$. First, $y(t)$ touches the graph when $t = \frac{1}{\beta}\left(n+\frac{1}{2}\right)\pi = \frac{10}{\sqrt{9999}}\left(n+\frac{1}{2}\right)\pi$ for some integer n. But

$$\frac{10}{\sqrt{9999}}\arctan\sqrt{9999} \neq \frac{10}{\sqrt{9999}}\left(n+\frac{1}{2}\right)\pi$$

for any integer n because $\arctan x$ is always strictly between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. But by using a computer, the difference between the left-hand side and right-hand side when $n = 0$ is about:

$$0.001000066672...$$

so the maximum value is very close to where the graph of $y(t)$ touches the graph of $\frac{1}{\beta}e^{-\alpha t}$.