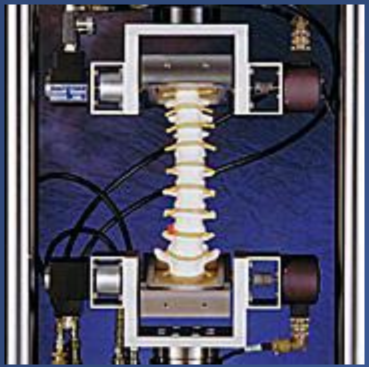


Biomechanics Fundamentals

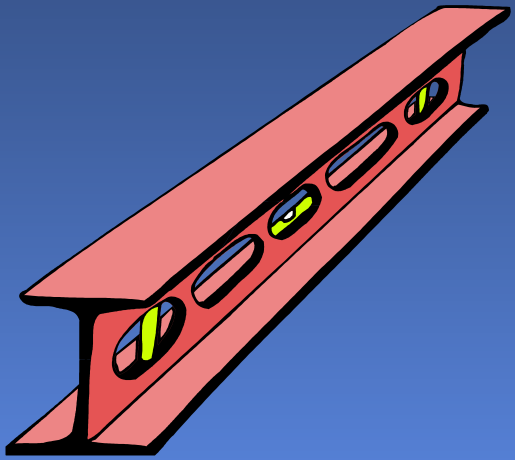
A *Very* Short Course

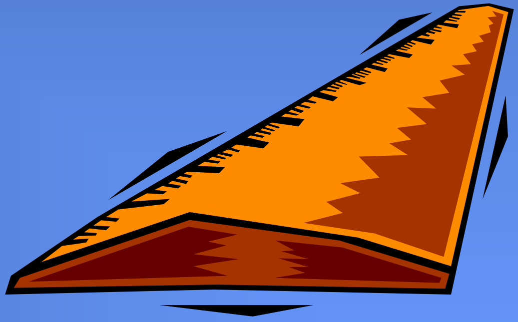
Objectives

- What is a *Material Property*
- Practical testing is based on *Continuum Mechanics*
- Stress – Strain relationship
  - Tensor quantities
- Elasticity
  - Viscoelasticity
  - Poro-viscoelasticity
- Testing Approaches
  - Uniaxial
  - Biaxial
  - Dynamic

Material Property

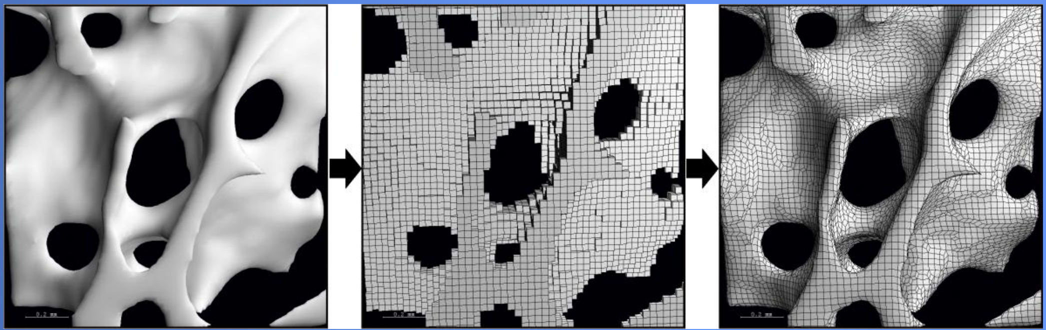
- A material property is a quantity that describes a physical attribute of a material, ***<u>independent</u>*** *of the shape or geometry of the material*
  - *Examples*
    - *Modulus of elasticity*
    - *Poisson's ratio*
- *System Property examples*
  - *Stiffness*
  - *Compliance (the inverse of stiffness)*





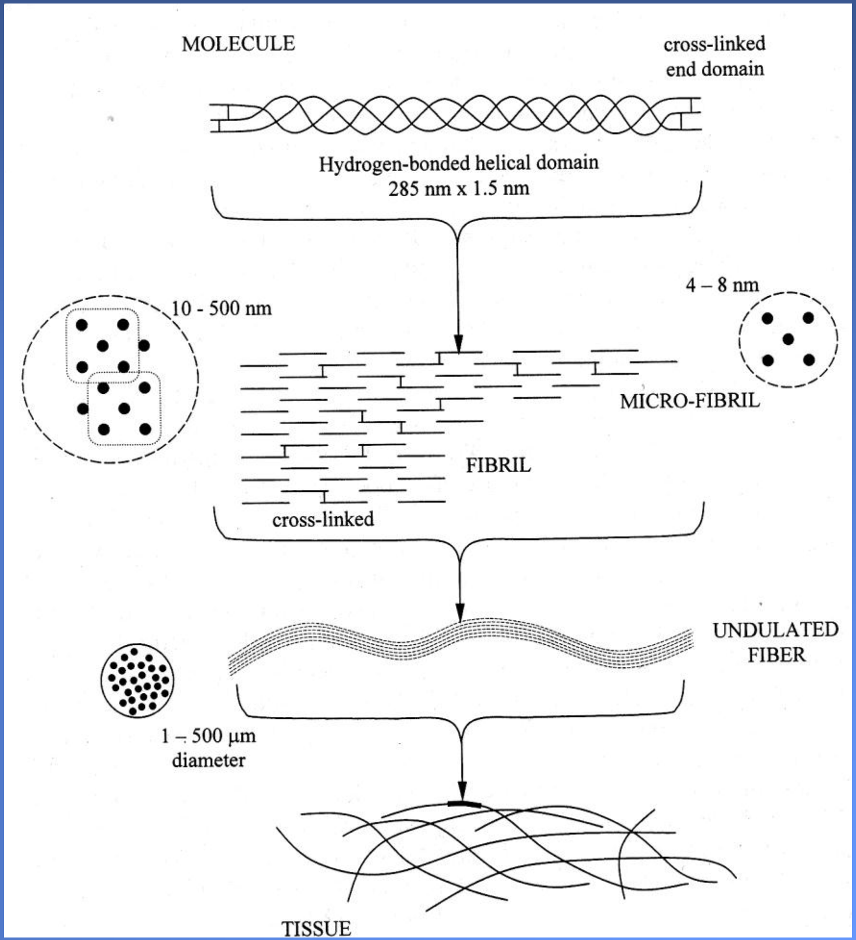
Continuum Mechanics

- We ignore the underlying physics of the smallest particles to obtain relevant information of the system of interest
- Is based on the *assumption* that $\delta/\lambda \ll 1$
  - □ δ = characteristic length scale of the microstructure
  - □ λ = characteristic length scale of the physical problem of interest
- Example: interested in loading of cells in wall of a large artery
  - $\delta \sim$ micrometers (μm)
  - $\lambda \sim$ millimeters (mm)
  - $\delta/\lambda = 10^{-6}/10^{-3} = 10^{-3} = 0.001 \ll 1$



Continuum Mechanics (II)

- Consider collagen based tissues (e.g., skin)
  - Collagen fiber has a diameter on order of ~1 μm (~10^{-6} m)
  - Hence, continuum approach is reasonable for tissues with dimensions on order of ~10^{-3} m (1 mm) or greater



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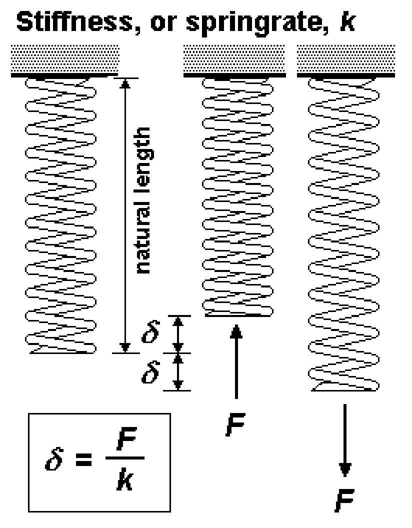
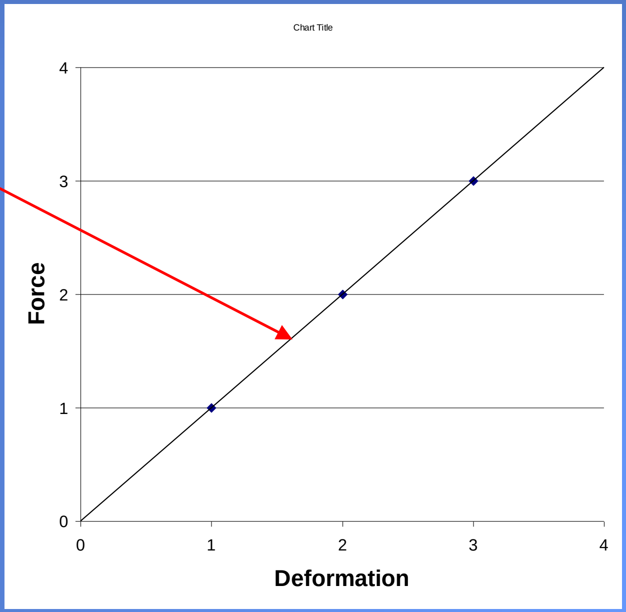
Stiffness

- Consider a spring that is loaded in compression or tension
  - Applying a force (F) results in a deformation (δ)
  - The stiffness (k) is the ratio of the force/deformation
    - The slope of the line is a graphical depiction of the magnitude of <u>*k*</u>
    - $$F = k(x - x_0)$$
    - Where x is the current position, and x_0 is the original position





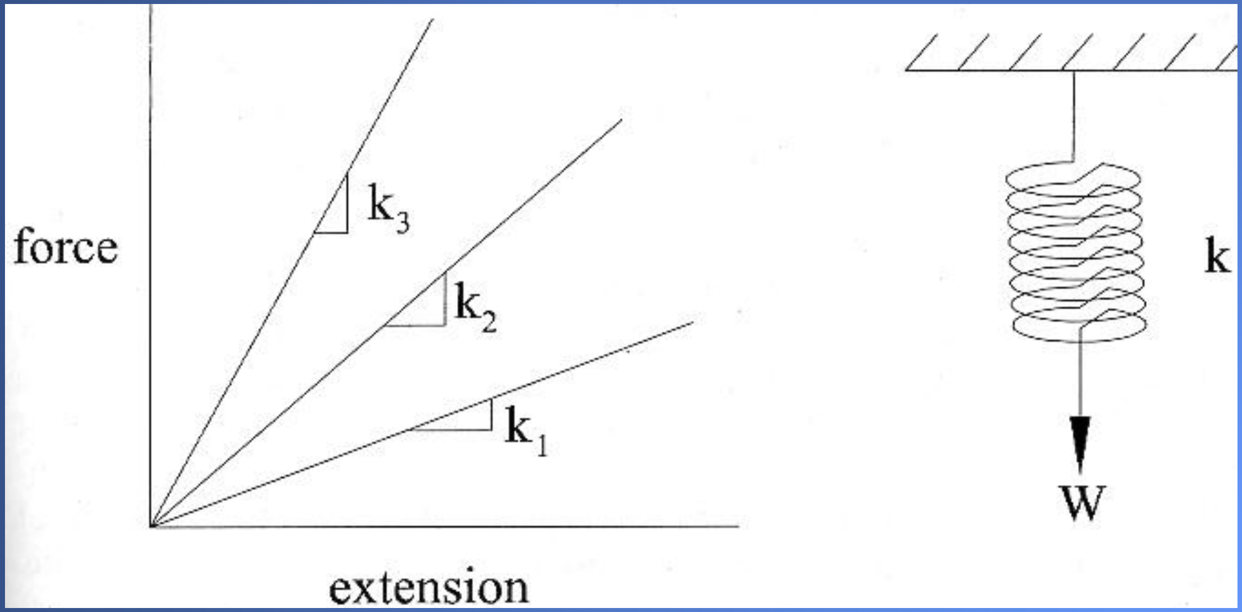
Stiffness

- Robert Hooke, in 1678, published the seminal observation (in latin)
  - *ut tension sic vis*
  - which can be translated as, 'as the force, so the extension'
- Hooke is attributed with "Hooke's Law"
  - Linear relationship between stress & strain
  - However, this was not his doing

http://www.edu365.com/aulanet/comsoc/persones_tecniques/Robert_Hooke_archivos/Robert_Hooke.jpg

Difference in Spring Constants

- The graph in the figure depicts the basic relationship between force and deformation for springs with three different "stiffness" values (i.e., k1 < k2 < k3)



Humphrey & Delange, 2004

Historical Development of Stress

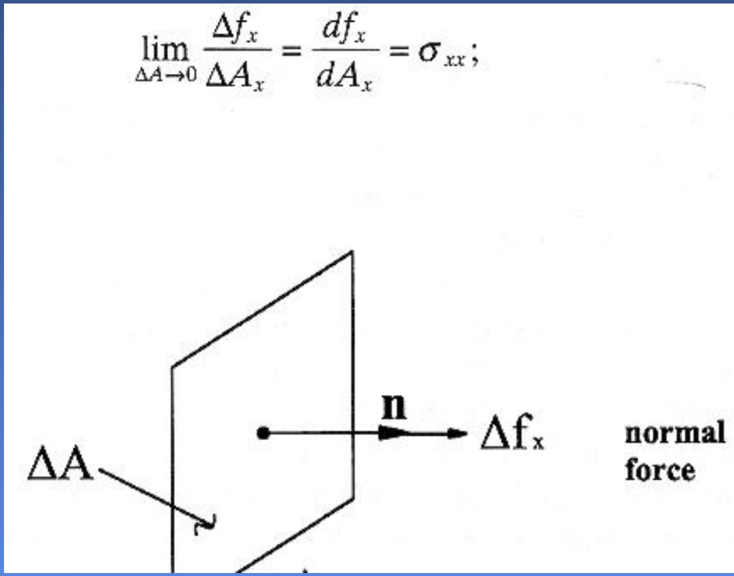
- Leonard Euler (pronounced "*Oil-er*") in 1757 (almost 80 years after Hooke) developed the seminal definition of stress
  - He called it "force intensity"
  - Defined for a force acting perpendicular to an area of interest
- Augustin-Louis Cauchy in 1827 (50 years after Euler) developed the basis of our modern understanding of stress
  - Stress is defined for an oriented area
  - For a given differential volume (which in the limit is a point), the magnitude of stress varies based on the orientation of the area of interest

http://www.herder-oberschule.de/images/euler.jpg

http://www-gap.dcs.st-and.ac.uk/~history/BigPictures/Cauchy_5.jpeg

Normal Stress Definition

- Stress is dependent upon the choice of coordinate systems (which is defined by an origin and “basis”)
- Consider a differential force (D*f*) that is perpendicular (or “normal”) to a differential area (DA)
- As it is an arbitrary choice, we choose to define this force (D*f*) as oriented along the X axis, and hence call it D*f*x
- We then define the normal stress as the following limit:



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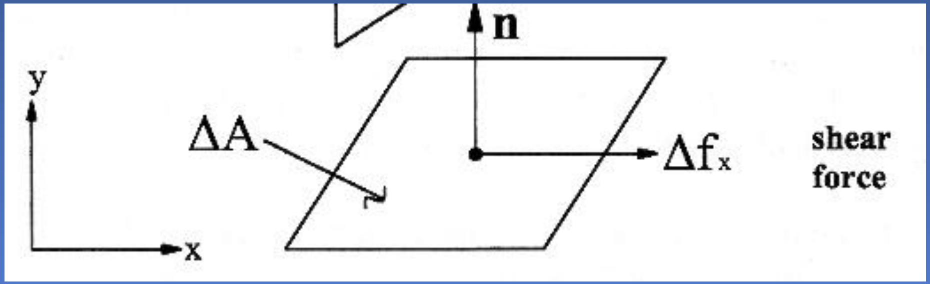
$$\sigma_{xx} \equiv \lim_{\Delta A \to 0} \frac{\Delta f_x}{\Delta A_x} = \frac{df_x}{dA_x}$$

Shear Stress

- It is also possible to define a force that would be oriented parallel to the same differential area.
- More formally, we state that the force is perpendicular to the unit outward normal vector ***n*** (where |**n**| = 1).
- This situation is called a *shear stress* and is defined as the following limit:

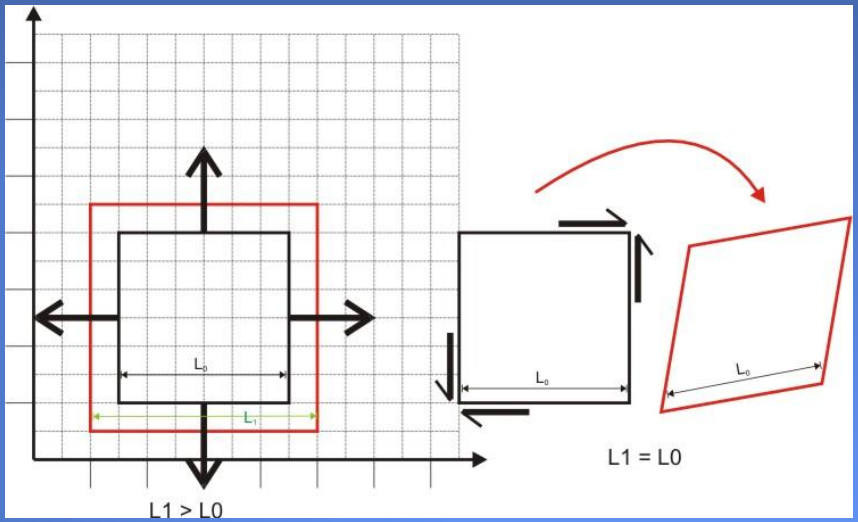


Humphrey & Delange, 2004

$$\sigma_{yx} \equiv \lim_{\Delta A \to 0} \frac{\Delta f_x}{\Delta A_y} = \frac{df_x}{dA_y}$$

Conceptual differences between normal and shear stresses

- “Pure” or uniform tension (or compression) results in a change in volume but not shape (i.e., if the material was a cube to start, it would still be a cube after uniform tension (dilational stress)
- Pure shear results in a change of shape but not of volume (deviatoric stress)

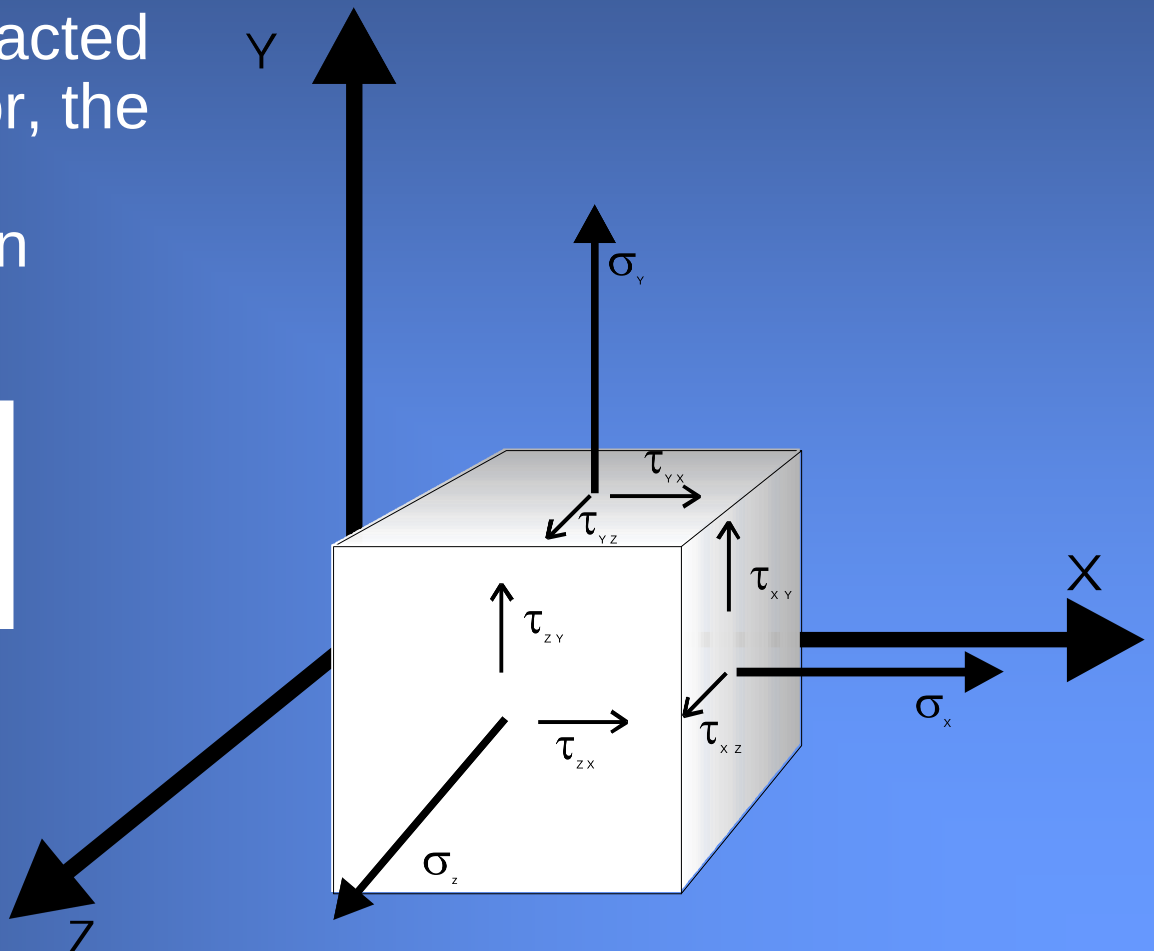


General Stress Equation

- For an infinitesimal volume (which we can think of as a teeny tiny little cube) acted upon by a force vector, the force vector can be described in Cartesian coordinates as

$$\Delta\bar{f} = \Delta f_x \hat{i} + \Delta f_y \hat{j} + \Delta f_z \hat{k}$$

$$\equiv \Delta f_x \hat{e}_x + \Delta f_y \hat{e}_y + \Delta f_z \hat{e}_z$$



Stress Tensor

- three different ways to represent the nine (9) stress components
- subscripts to represent the X, Y, Z axes
- *numerical indices*, which is also known as *Einsteinian notation*
  – indices i & j can take values of only 1, 2, and 3 (i.e., i = {1,2,3} and j = {1,2,3})

$$\boldsymbol{\sigma} = \begin{bmatrix} \dfrac{df_x}{dA_x} & \dfrac{df_y}{dA_x} & \dfrac{df_z}{dA_x} \\ \dfrac{df_x}{dA_y} & \dfrac{df_y}{dA_y} & \dfrac{df_z}{dA_y} \\ \dfrac{df_x}{dA_z} & \dfrac{df_y}{dA_z} & \dfrac{df_z}{dA_z} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Tensor Definition

- Stress is NOT a vector quantity, but rather admits to a **tensor** form.
- Tensors have an ‘order’ or ‘rank’,
  - Stress (as well as strain) is a 2nd order (or rank) tensor.
  - Vectors can be shown to be a 1st order tensor
  - Scalars are a 0th order tensor

Stress tensor details

- The terms on the diagonal are the *normal stresses* and the off-diagonal terms are *shear stresses*.
  - Normal stresses are:
    - $\sigma_{ii} = (\sigma_{11}, \sigma_{22}, \sigma_{33})$
  - Shear stresses are
    - σij ($\sigma_{12}, \sigma_{13}, \sigma_{23}$).
- If the system is in equilibrium, then
  - the shear stresses must have certain equivalencies
  - Hence: $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, $\sigma_{23} = \sigma_{32}$.
- Thus, although there are fully nine (9) components of the stress tensor, there are **<u>only</u>** six (6) independent components

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Tensors are not dependent on choice of coordinate system

- Coordinate systems other than the Cartesian system can also be used, depending upon what is more convenient for the physical system.

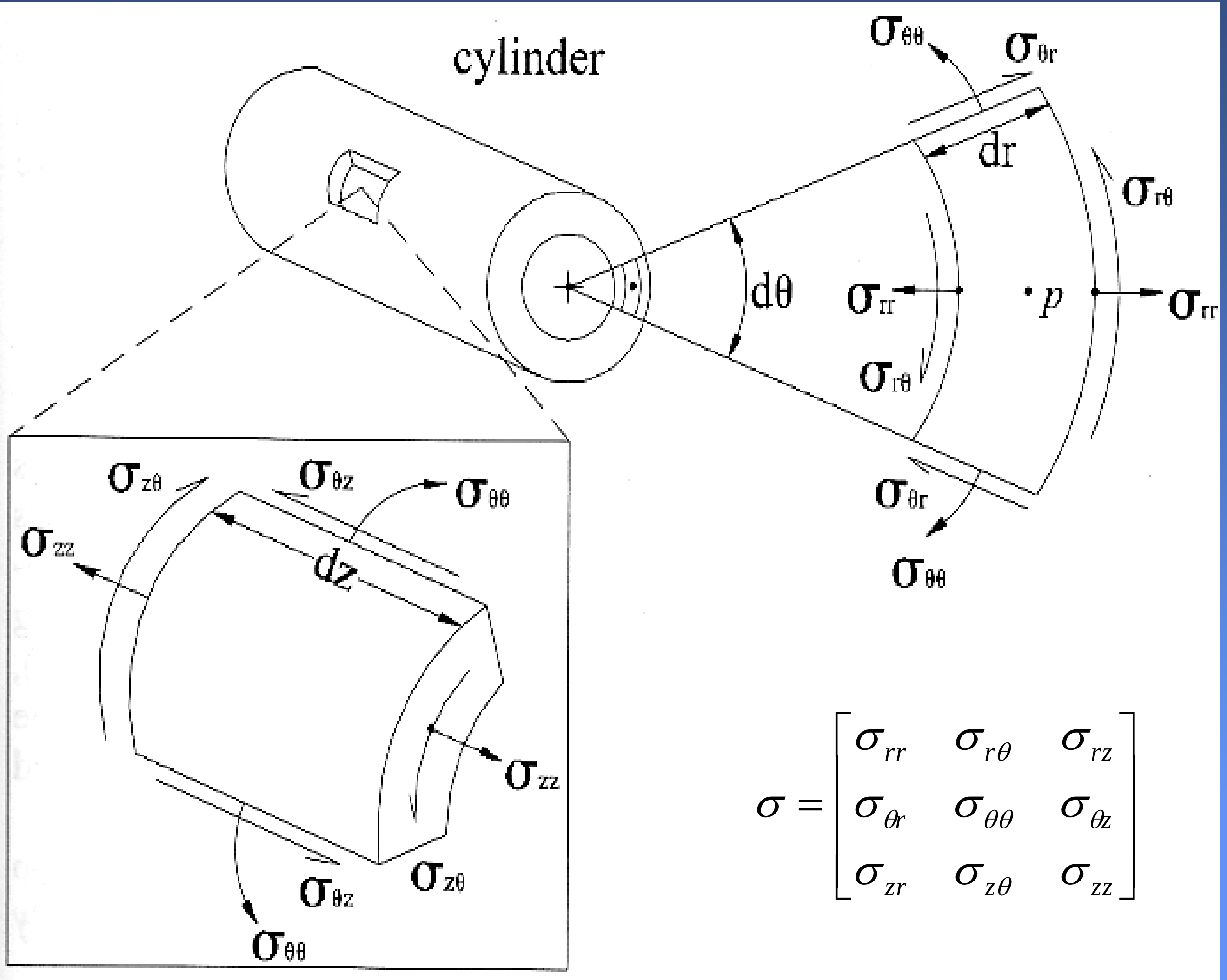
Cylindrical

$$\sigma = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$$

Spherical

$$\sigma = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix}$$

Cylindrical Stresses



$$\sigma = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$$

Humphrey & Delange, 2004

Spherical Stresses

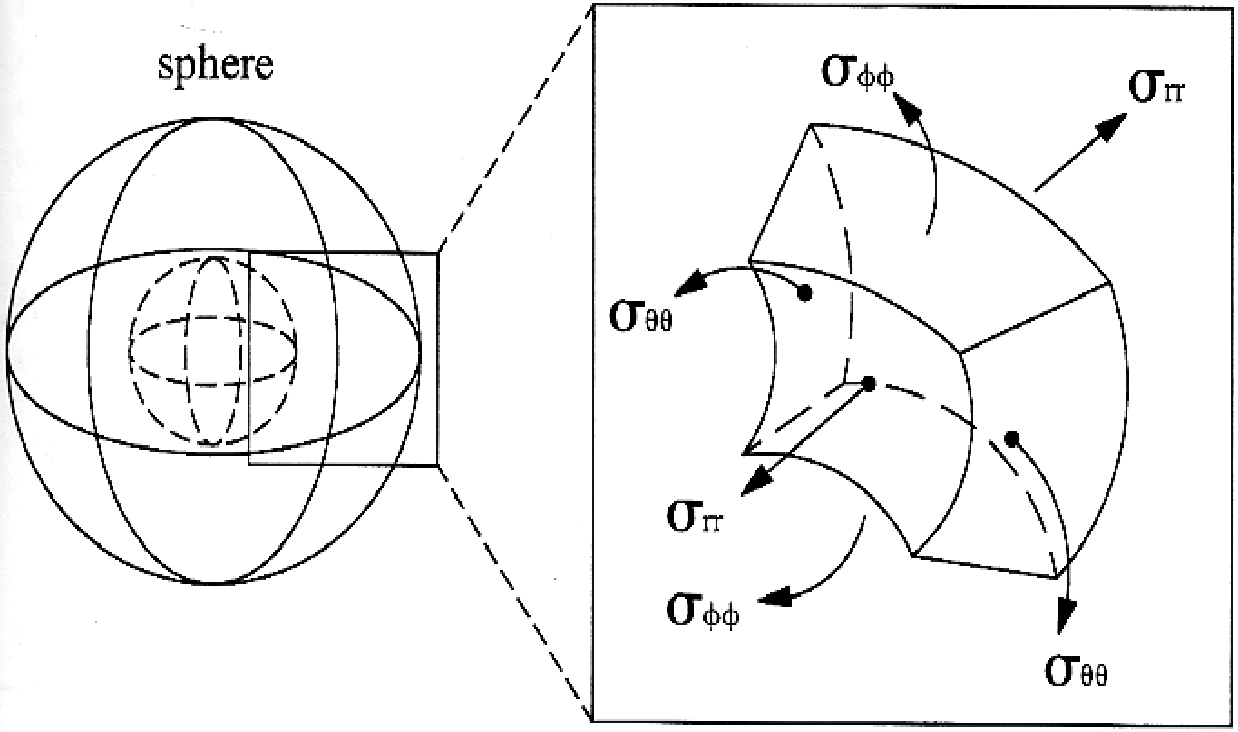


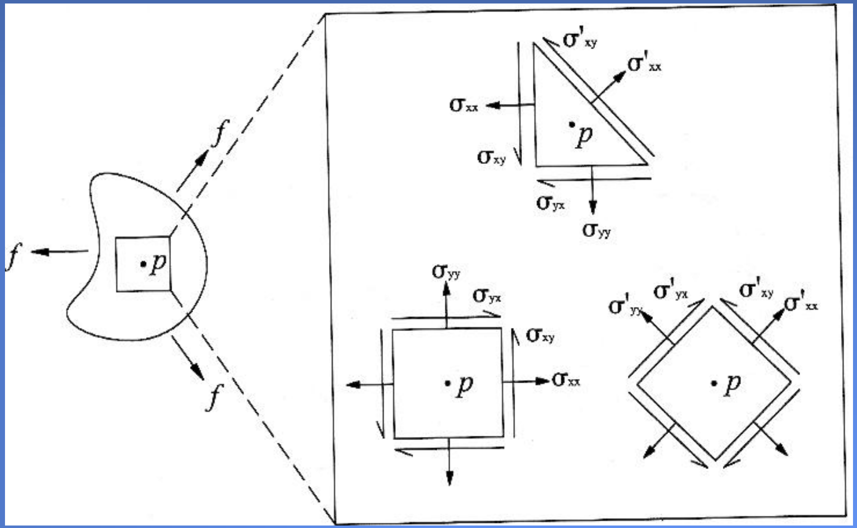
FIGURE 2.9 Normal components of stress relative to a spherical coordinate system, again denoting the components as $\sigma_{(\text{face})(\text{direction})}$. As an exercise, add the shearing components.

$$\sigma = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix}$$

Humphrey & Delange, 2004

Stress Transformations

- Consider a 2-D state of stress relative to
  - $o; \hat{e}_x, \hat{e}_y$
  - $o; \hat{e}'_x, \hat{e}'_y$
- What we seek is a method by which we can determine the stress for any orientation of interest, which could facilitate our understanding of the stresses that develop in a tissue.



2D Stress Transformation

$$\sigma_{xx'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos(2\alpha) + \sigma_{xy}\sin(2\alpha)$$

$$\sigma_{yy'} = \frac{\sigma_{yy}+\sigma_{xx}}{2} - \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos(2\alpha) - \sigma_{xy}\sin(2\alpha)$$

$$\sigma_{x'y'} = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin(2\alpha) + \sigma_{xy}\cos(2\alpha)$$

Principal Stress

- "Is there some angle, α_p, such that the shear stress is identically zero (0)?"
- To find the min or max of any equation, we take its derivative & set it equal to 0

$$\sigma'_{xx} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\alpha + \sigma_{xy}\sin 2\alpha$$

$$\frac{d\sigma'_{xx}}{d\alpha} = 0 = \frac{\sigma_{xx} - \sigma_{yy}}{2}(-\sin 2\alpha)(2) + \sigma_{xy}(\cos 2\alpha)(2)$$

$$\frac{\sin 2\alpha_p}{\cos 2\alpha_p} = \frac{\sigma_{xy}}{(\sigma_{xx} - \sigma_{yy})/2} = \tan 2\alpha_p$$

Principal Angle

$$\alpha_p = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{xy}}{(\sigma_{xx} - \sigma_{yy})}\right)$$

2D Principal Stresses

- The non-zero stresses at the principal angle are the principal stresses

$$\sigma_1 \equiv \sigma'_{xx\downarrow_{\max/\min}} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

What is Pressure?

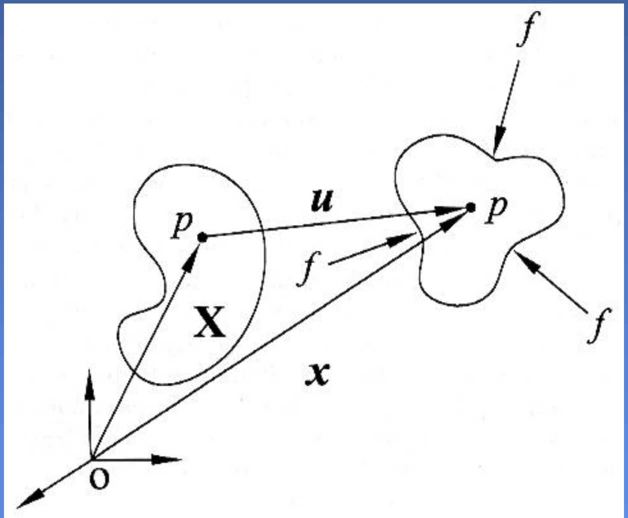
- **Pressure** is the mathematically trivial state of stress where the normal (or axial) stresses are all the same for any transformation (or rotation) of coordinate systems
  - Given the following 2-D stress
    - $\sigma_{xx} = \sigma_{yy} = -p$; $\sigma_{xy} = 0$

$$\sigma'_{xx} = \sigma_{xx}\cos^2\alpha + 2\sigma_{xy}\sin\alpha\cos\alpha + \sigma_{yy}\sin^2\alpha$$

$$\begin{aligned}\sigma'_{xx} &= -p\cos^2\alpha + 2(0)\sin\alpha\cos\alpha + -p\sin^2\alpha \\ &= -p\left(\cos^2\alpha + \sin^2\alpha\right) + 0 \\ &= -p \quad \forall\alpha\end{aligned}$$

Strain

- *In solid biomechanics, the basis of strain determination is the displacement* vector (**u**) for a point
  - *current position is denoted by a position vector **x***
  - original position is denoted by a position vector **X**
  - Thus, **u = x – X**
- *Displacement vectors can also vary with time, as well as position.*
  - *Hence: u(**X**, t) = **x**(**X**, t) – **X***



Humphrey & Delange, 2004

$$\bar{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k} \equiv u_x\hat{e}_x + u_y\hat{e}_y + u_z\hat{e}_z$$

Lagrangian Strain Tensor

$$\gamma_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j} \right)$$

- Where *i* and *j* take values of {1, 2, 3}
- The above notation is a compact form of the strain tensor

Strain tensor (cont'd)

$$\gamma_{ij} = E_{ij} = \begin{Bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix} = \begin{Bmatrix} E_{XX} \\ E_{YY} \\ E_{ZZ} \\ E_{XY} \\ E_{YZ} \\ E_{XZ} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_x}{\partial X} + \frac{1}{2}\left(\left(\frac{\partial u_x}{\partial X}\right)^2 + \left(\frac{\partial u_Y}{\partial X}\right)^2 + \left(\frac{\partial u_Z}{\partial X}\right)^2\right) \\ \frac{\partial u_Y}{\partial Y} + \frac{1}{2}\left(\left(\frac{\partial u_X}{\partial Y}\right)^2 + \left(\frac{\partial u_Y}{\partial Y}\right)^2 + \left(\frac{\partial u_Z}{\partial Y}\right)^2\right) \\ \frac{\partial u_Z}{\partial Z} + \frac{1}{2}\left(\left(\frac{\partial u_X}{\partial Z}\right)^2 + \left(\frac{\partial u_Y}{\partial Z}\right)^2 + \left(\frac{\partial u_Z}{\partial Z}\right)^2\right) \\ \frac{1}{2}\left(\frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X} + \frac{\partial u_X}{\partial X}\frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X}\frac{\partial u_X}{\partial Y} + \frac{\partial u_Z}{\partial X}\frac{\partial u_Z}{\partial Y}\right) \\ \frac{1}{2}\left(\frac{\partial u_Y}{\partial Z} + \frac{\partial u_Z}{\partial Y} + \frac{\partial u_x}{\partial Y}\frac{\partial u_X}{\partial Z} + \frac{\partial u_Y}{\partial Y}\frac{\partial u_Y}{\partial Z} + \frac{\partial u_Z}{\partial Y}\frac{\partial u_Z}{\partial Z}\right) \\ \frac{1}{2}\left(\frac{\partial u_Z}{\partial X} + \frac{\partial u_X}{\partial Z} + \frac{\partial u_x}{\partial X}\frac{\partial u_X}{\partial Z} + \frac{\partial u_Y}{\partial X}\frac{\partial u_Y}{\partial Z} + \frac{\partial u_Z}{\partial X}\frac{\partial u_Z}{\partial Z}\right) \end{Bmatrix}$$

Where: $\gamma_{12} = \gamma_{21}, \gamma_{23} = \gamma_{32}, \gamma_{13} = \gamma_{31}$

Green's strain tensor represents *nonlinear* strain- displacement relations in *Cartesian* coordinates

Small strain simplification

- If the displacement is small,
  - then the displacement gradients are small,
  - then (& only then) the nonlinear terms (i.e., the squared terms) in the Green's strain equations can be ignored without significant loss of accuracy.
  - That is, if $u_{i,j} >> (u_{i,j})^2$, then the small strain formulation can be used, which is as follows

Small strain simplification

- How small is *small*?
- General guideline is that the small strain approximation can be used for strains up to 5% (0.005)

$$[\varepsilon] = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_z}{\partial z} \\ \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_z}{\partial y} + \dfrac{\partial u_y}{\partial z}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z}\right) \end{Bmatrix}$$

Small strain calculation in 2D

- To calculate the strain, we simply take the partial differentials. As this is only a 2D case, we can ignore the out-of-plane components. Let’s do it, first for the linearized formulation

$$[\varepsilon] = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_x}{\partial X} \\ \dfrac{\partial u_y}{\partial Y} \\ \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial Y} + \dfrac{\partial u_y}{\partial X} \right) \end{Bmatrix}$$

Strain tensor transformation

- Strain, like stress, is based on an arbitrary choice of coordinate systems
- Strain tensors can thus be transformed
- Principal strains are the non-zero strains that remain once the coordinate system is rotated such that the shear strains go to zero

$$\varepsilon_1 = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \varepsilon_{xy}^2}$$

$$\varepsilon_2 = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} - \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \varepsilon_{xy}^2}$$

Relationship between stress and strain

- The relationship between stress and strain defines the *constitutive behavior* of a material.
- It is independent of the geometry of the material. Why?
  - Because stress is the force normalized by the relevant cross-sectional area and strain is already a non-dimensional quantity (displacement change divided by original length).
- Hence, the relationship between stress and strain depends on the properties intrinsic to the material itself, which are appropriately called material properties

Hook's Law

- $\sigma = C\varepsilon$
- The 'C' term in this equation is a matrix (stifness matrix),
  - given that both the stress and strain tensors have 9 terms,
  - then the C matrix must be a 9 x 9 matrix, or have 81 terms.
- However, in equilibrium, we know that we can reduce the stress and strain tensors to only 6 independent terms,
- hence, this reduces the size of the C matrix to a total of: 6 x 6 = 36 terms

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45`} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix}$$

C matrix is symmetrical, such that $C_{12} = C_{21}$, $C_{13} = C_{31}$, etc., and hence, the 36 independent terms are reduced to 21 independent terms

Linear isotropy

- If the material is linear and also *isotropic,*
  - then the C matrix reduces to one with only two independent quantities
    - called the Lame constants
      - G
      - μ

$$\mu = C_{12} = C_{13} = C_{23}$$

$$G = C_{44} = C_{55} = C_{66}$$

$$\mu + 2G = C_{11} = C_{22} = C_{33}$$

Poisson's ratio, & Lame constants

- It can be shown (i.e., we're not developing it here) that ,
  - where E is the so-called Young's modulus,
  - and ν is Poisson's ratio, which is defined as the negative ratio of the orthogonal strain to an applied strain.
- The importance of these latter two observations are that, if the material is linear isotropic, then, the <u>entire material properties</u> can be determined by performing some rather simple testing to determine E and ν

$$G = \frac{E}{2(1+\nu)}$$

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}}$$

$$\mu = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

Linear, Elastic, Homogeneous, and Isotropic (LEHI)

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right)$$

$$\varepsilon_{22} = \frac{1}{E}\left(\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right)$$

$$\varepsilon_{33} = \frac{1}{E}\left(\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right)$$

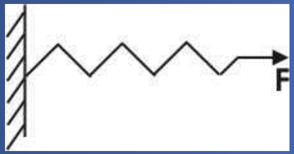
$$\varepsilon_{12} = \frac{\sigma_{12}}{2G}$$

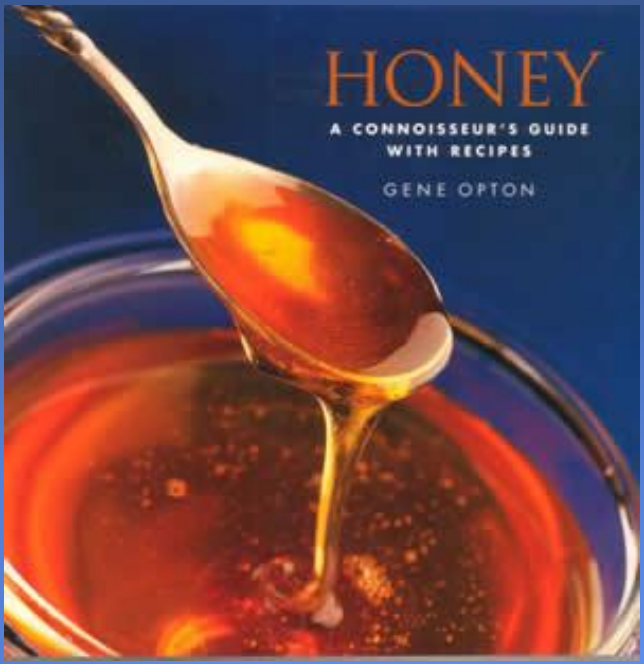
$$\varepsilon_{13} = \frac{\sigma_{13}}{2G}$$

$$\varepsilon_{23} = \frac{\sigma_{23}}{2G}$$

Viscoelasticity

- to develop procedures to determine the elastic and viscous moduli used in various viscoelastic models relating stress and strain
- a spring element to represent the stiffness of a linearly elastic material

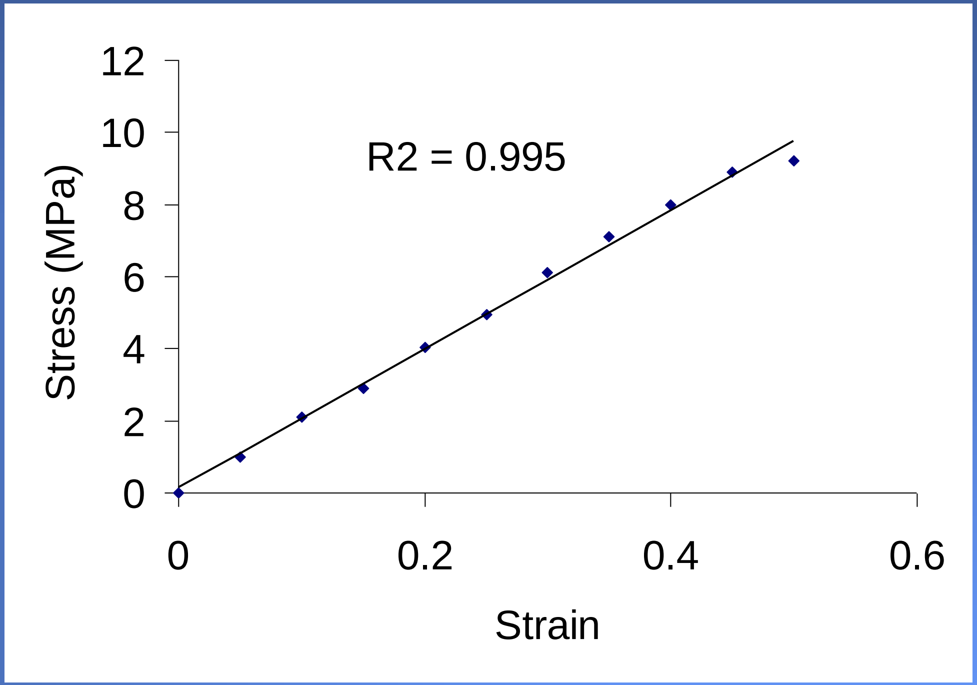




http://www.beedata.com/nbb/honey-opton.jpg

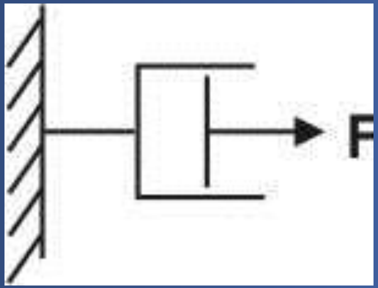
Elasticity in Viscoelasticy

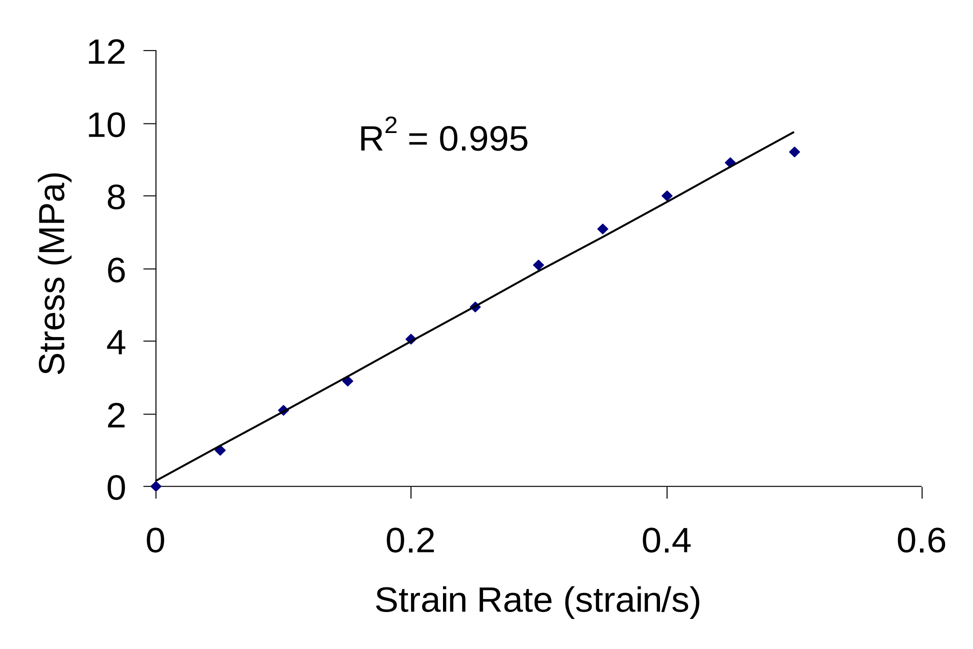
- The modulus, E, of the spring is determined by applying various strains and measuring the developed stress.
- The data points are then regressed or curve fit to determine the slope of the line that best represents the data.
- The slope of the line is the measure of the modulus of elasticity or stiffness



Viscous in Viscoelasticity

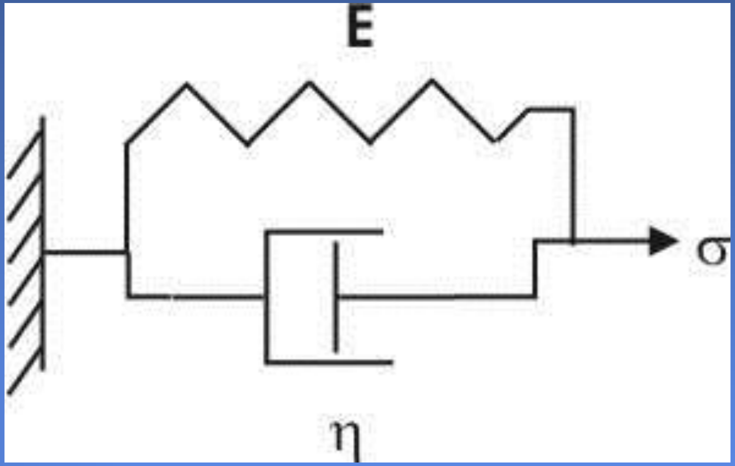
- Depicting a viscous material by using a dashpot element





Empirical models of viscoelasticity

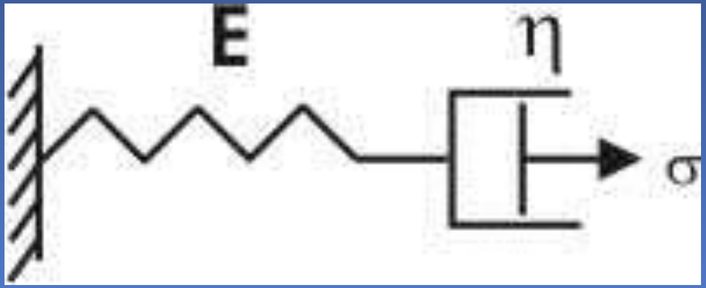
- Kelvin-Voight Model
- This model represents a viscoelastic solid and is depicted by placing a spring and dashpot elements in parallel
- Stress – strain relationship for this model is expressed as:



$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

Maxwell Model

- This model represents a viscoelastic fluid. It is depicted by a spring and dashpot in series
- 2 parameter equation relating stress and strain is

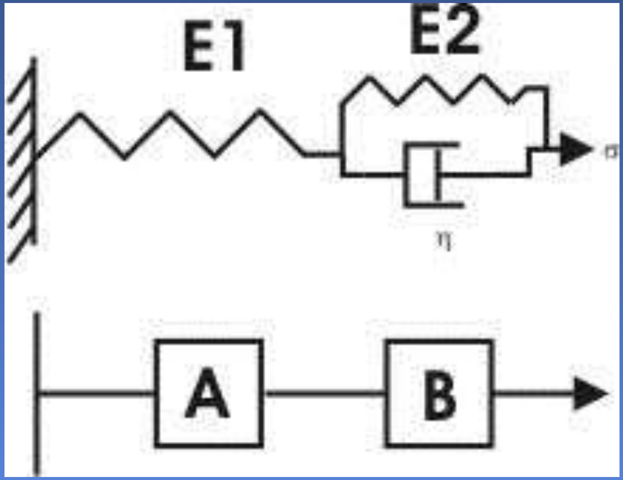


$$\eta\dot{\sigma} + E\sigma = E\eta\dot{\varepsilon}$$

$$\sigma = \frac{\eta}{E}\left(E\dot{\varepsilon} - \dot{\sigma}\right)$$

Standard solid Model

- reasonable estimate of the stress strain behavior of a number of real biological materials
  - cartilage
  - white blood cell membrane
- 3 parameter (E1, E2, η) function is



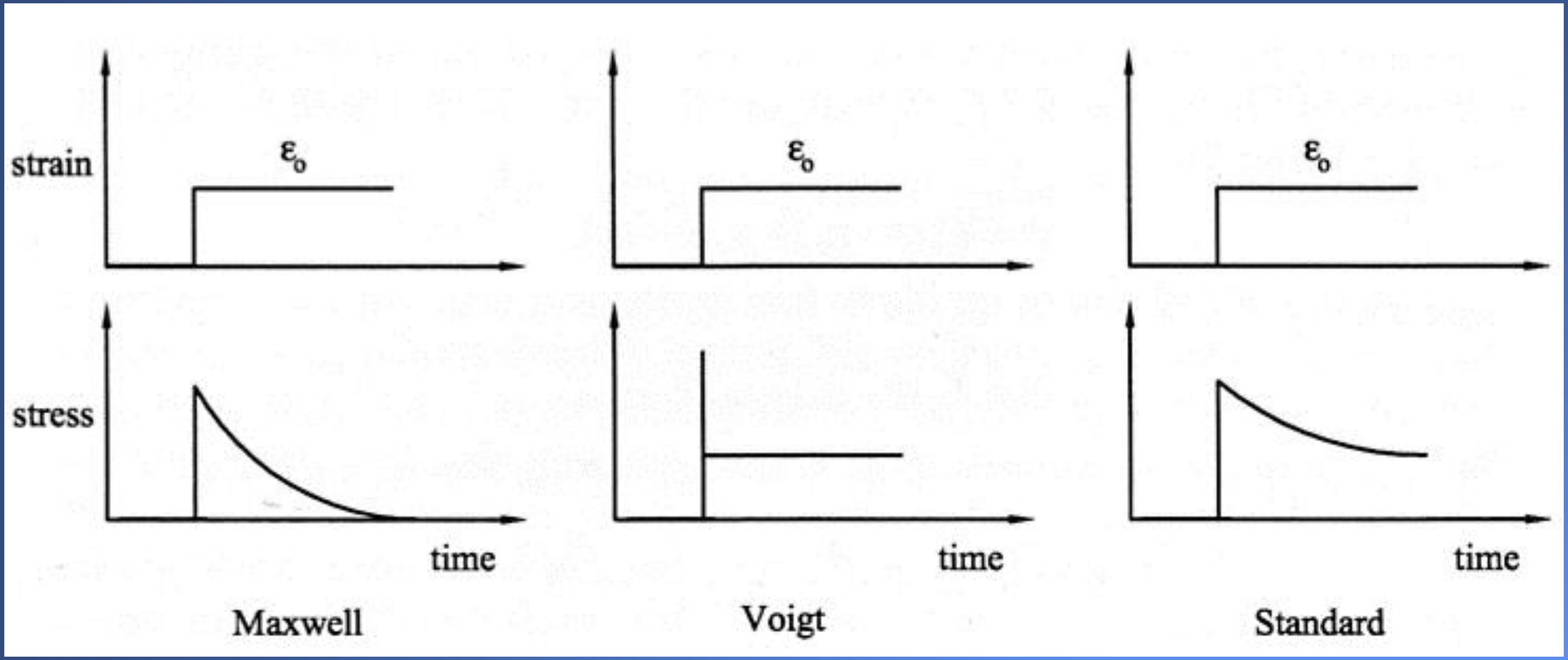
$$(E_1 + E_2)\sigma + \eta\dot{\sigma} = E_1 E_2 \varepsilon + E_1 \eta \dot{\varepsilon}$$

$$\sigma = \frac{E_1 E_2 \varepsilon + \eta(E_1 \dot{\varepsilon} - \dot{\sigma})}{E_1 + E_2}$$

Creep & Relaxation

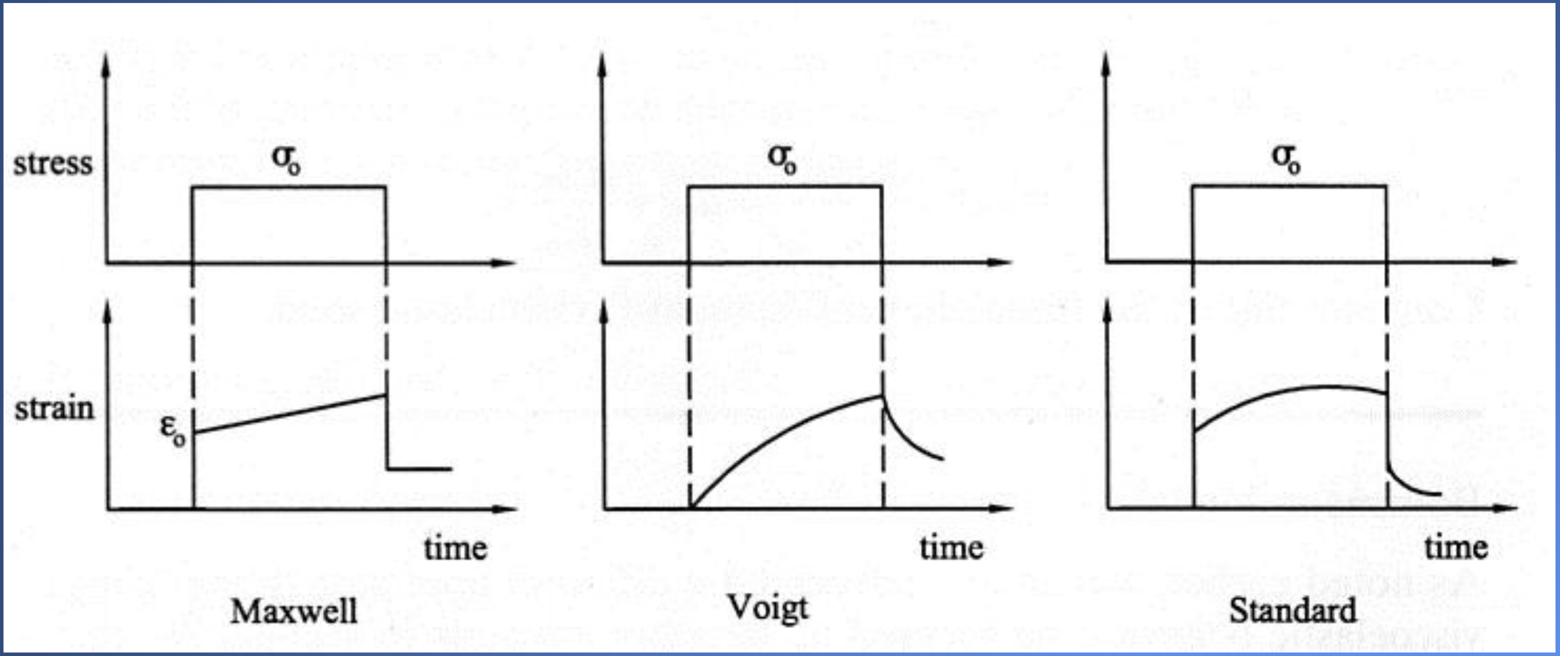
- Empirically, we observe that if the strain is held constant, then viscoelastic tissues will demonstrate a “stress relaxation”, where the stress decreases until it reaches a new equilibrium value.
- Inversely, if the stress is held constant, then viscoelastic tissues will demonstrate a phenomenon called “creep”, where the strain increases until it reaches an equilibrium value
- Is glass viscoelastic?

Model Comparisons – Stress Relaxation



Characteristic stress relaxation responses
of the Maxwell, Kelvin-Voigt, and Standard element

Creep Responses



Characteristic creep responses for the Maxwell, Kelvin-Voigt, and Standard element

Relaxation & Creep

- Define the *relaxation* function as *G(t)*, and the *creep* function as *J(t)*.

$$G(t) = \frac{\sigma(t)}{\varepsilon_0}, \quad J(t) = \frac{\varepsilon(t)}{\sigma_0}$$

- Maxwell model

$$G(t) = Ee^{Et/\mu}, \quad J(t) = \frac{1}{\mu}t + \frac{1}{E}$$

where μ is the viscosity and E is the Young's modulus

Standard Model – Stress Relaxation

- Standard Model

$$G(t) = E_0 + (E + E_0 - E_0)e^{-t/t_R} \equiv G_\infty + (G_0 - G_\infty)e^{-t/t_R}$$

  - where
    - $G_\infty \equiv E_0$
    - $G_0 \equiv E + E_0$
    - $t_R \equiv \mu / E$
  - The larger the value of t_R (via either a large μ or a small E), the slower the relaxation

Standard Model – Creep function

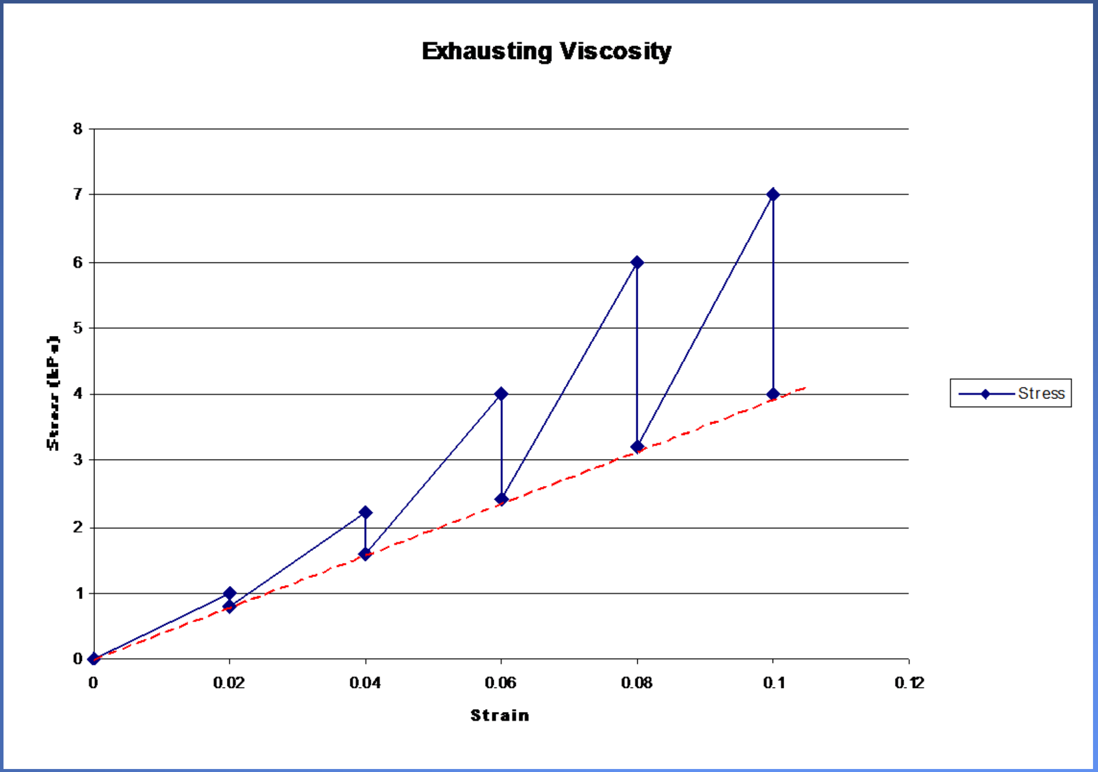
- Where
  - $J_0 = \dfrac{1}{G_0}$
  -
  - $J_\infty = \dfrac{1}{G_\infty}$

$$t_c = G_0 t_R / G_\infty$$

$$J(t) = J_\infty + (J_0 - J_\infty) e^{-t/t_c}$$

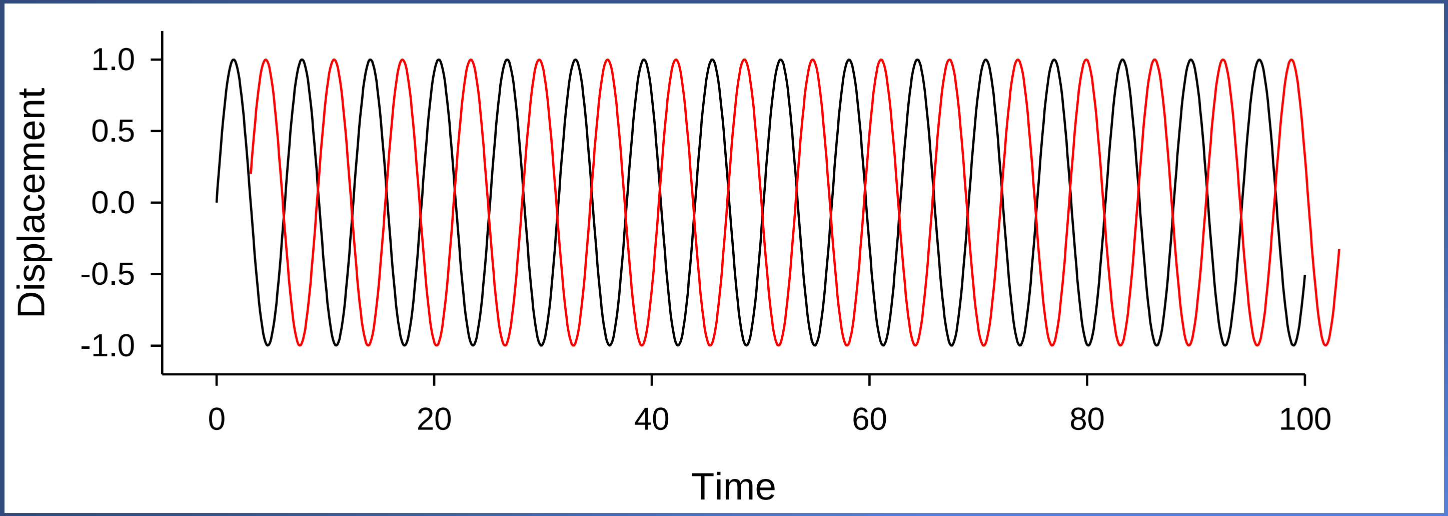
Measuring moduli

- How do we get the elastic & viscous moduli that make up these models?
- An approach for the Kelvin-Voight Model is



In this testing protocol, the material is strained to 2% and held until all the viscosity is exhausted (i.e., the tissue relaxes to equilibrium). Then, the tissue is strained another 2% (i.e., to 4%), until it reaches equilibrium. This process continues up to some maximum strain. The dashed red line represents the elastic component of the stiffness (i.e., E). The difference between the elastic component and the viscoelastic component represents the viscous modulus

Dynamic Testing



- Apply a sinusoidal tensile displacement (strain), resulting in a tensile stress
- We seek an expression that describes the observed phase lag in the stress to the strain.
- To do so, we utilize the Boltzman model (i.e., hereditary integrals).
- A priori, we suspect that there must be a term that accounts for some sort of elastic (or "storage") component and another term that accounts for "viscous" energy loss

Storage & Loss Moduli

- Boltzman form
  - We can denote the "storage" (in-phase, elasticity) and "loss" (out-of-phase, viscosity) moduli as
- Ratio of loss and storage moduli defines the phase angle φ

$$\sigma(t) = \varepsilon_A \left[G_1(\omega) \sin \omega t + G_2(\omega) \cos \omega t \right]$$

$$storage: G_1(\omega) \equiv \omega \int_0^\infty G(\tau) \sin(\omega \tau) d\tau$$

$$loss: G_2(\omega) \equiv \omega \int_0^\infty G(\tau) \cos(\omega \tau) d\tau$$

$$\tan \phi = \frac{G_2(\omega)}{G_1(\omega)}$$

Phase Lag

- For viscoelastic materials subjected to a strain history of the form , $\varepsilon(t) = \varepsilon_A \sin \omega t$
  - we expect there to be a phase lag between
    - stress
    - Strain
  - where ϕ
    - is the phase lag in radians

$$\sigma(t) = \sigma_A \sin(\omega t + \phi)$$

Energy relationship

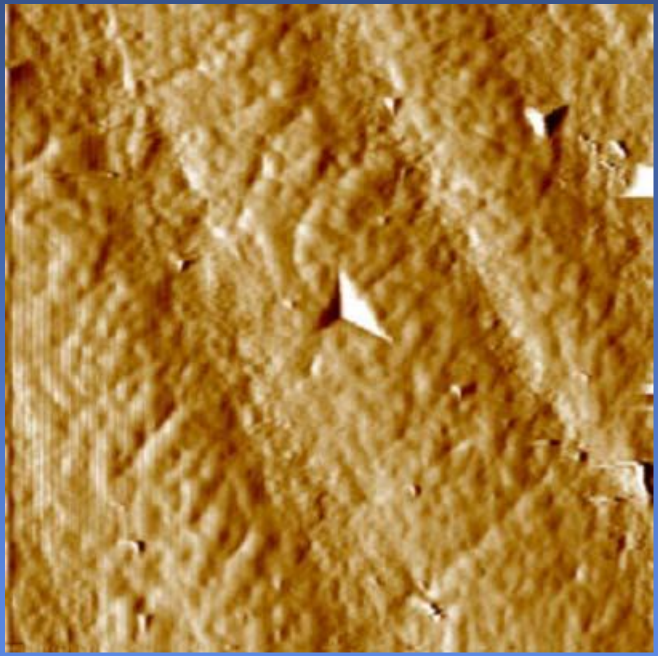
- Phase angle φ is often expressed as its tangent
  - called the viscous damping or *phase loss*.
- Tan φ is an indicator of the amount of strain energy lost relative to the energy stored per cycle
  - can be related to hysteresis
    - Resilience = 1 – hysteresis = 1 – 2 pi tan φ, when φ is small

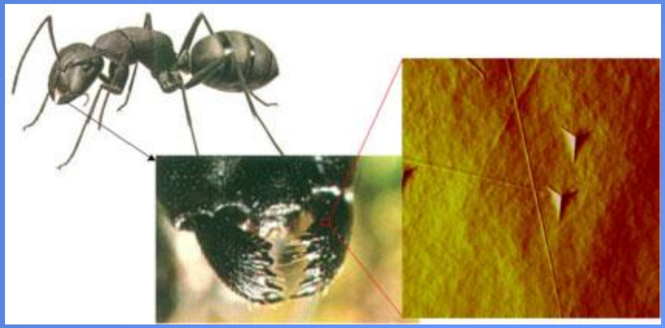
Estimation of Mechanical Properties from Biologic Tissues

- Conventional mechanical material testing of a larger structure
  (tension, compression, torsion, bending, constraints, rates, static, dynamic, fatigue)
- Mechanical testing at the micro- or nano-level
- Virtual mechanical testing (Finite Element Modeling, non-invasive determination of mechanical properties)

Nanoindentation

- Mechanical testing
  - Macro tests
  - Spatial variability is difficult to measure
- Nanoindentation
  - Elastic modulus
  - Hardness
  - Viscoelasticity





http://www.hysitron.com/Gallery/

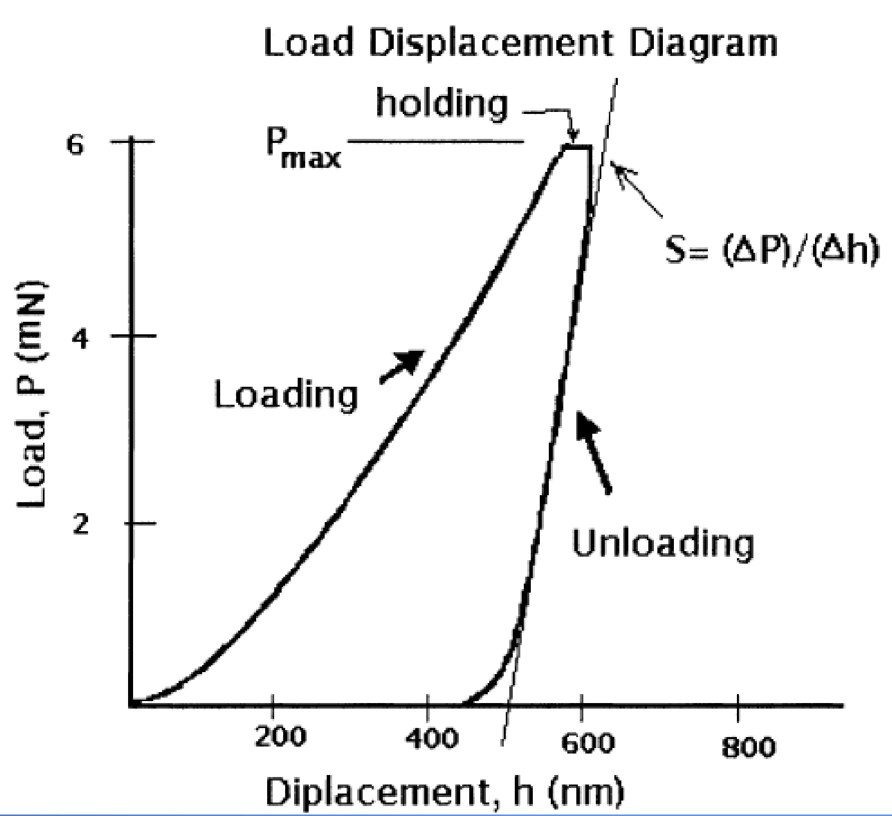
Interpretation of Indentation Data

- Typical load displacement curve of nanoindentation test
  - Contact stiffness (S) is calculated from the slope of unloading curve
  - Elastic modulus is calculated from S

$$E_r = \frac{\sqrt{\Pi}}{2} \frac{S}{\sqrt{A_c}}$$

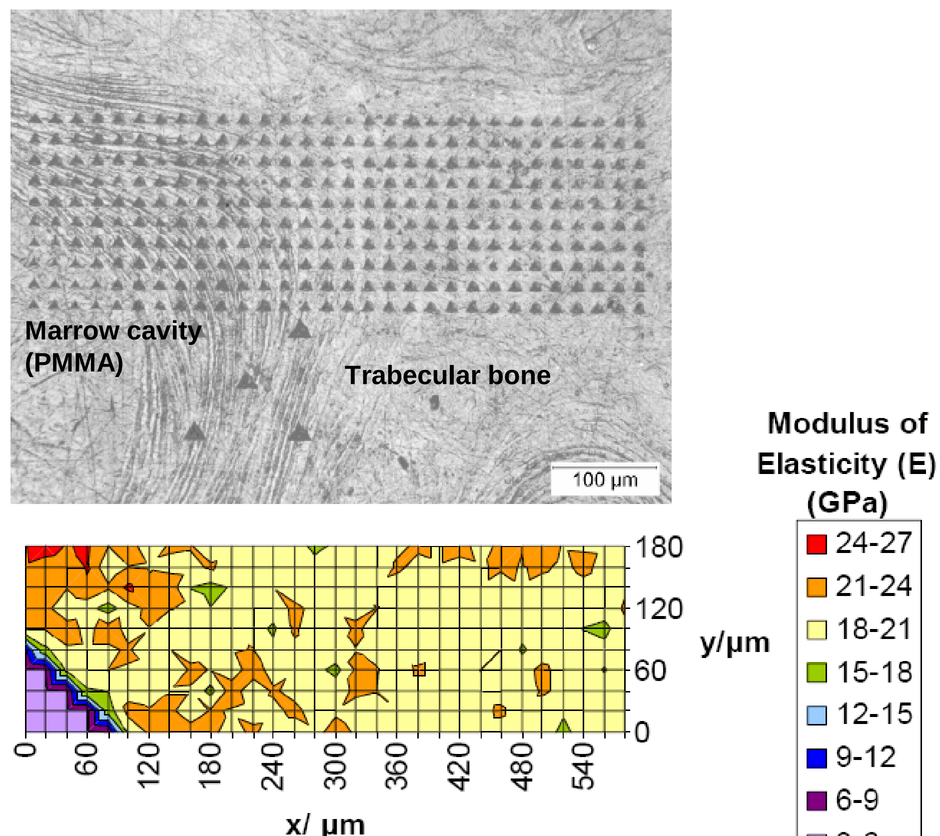
$$E_b = \frac{1-\nu_b^2}{\frac{1}{E_r} - \frac{1-\nu_i^2}{E_i}}$$

Busa et al., 2005



Akhter et al., 2003

Elastic Modulus within Human Mandibular Bone



Modulus Mapping of Array Area, 600 X 200 µm2

Ramey, Ayers & Ferguson