The Inner Product

Inner product or **dot product** of

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} :$$

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} =$$

$$u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Note that

$$\mathbf{v} \cdot \mathbf{u} = v_1 u_1 + v_2 u_2 + \cdots + v_n u_n$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \mathbf{u} \cdot \mathbf{v}$$

THEOREM 1

Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbf{R}^n$, and let c be any scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq \mathbf{0}$, and $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$.

Combining parts b and c, one can show

$$\boxed{(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{w} = c_1(\mathbf{u}_1 \cdot \mathbf{w}) + \cdots + c_p(\mathbf{u}_p \cdot \mathbf{w})}$$

Length of a Vector

For $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, the **length** or **norm of v** is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \qquad \text{and} \qquad \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}.$$

For example, if $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, then $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$ (distance between **0** and **v**)

Picture:

For any scalar c,

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$$

Distance in $\mathbf{R}^n$

The **distance between u and v** in $\mathbf{R}^n$:

$$\operatorname{dist}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|.$$

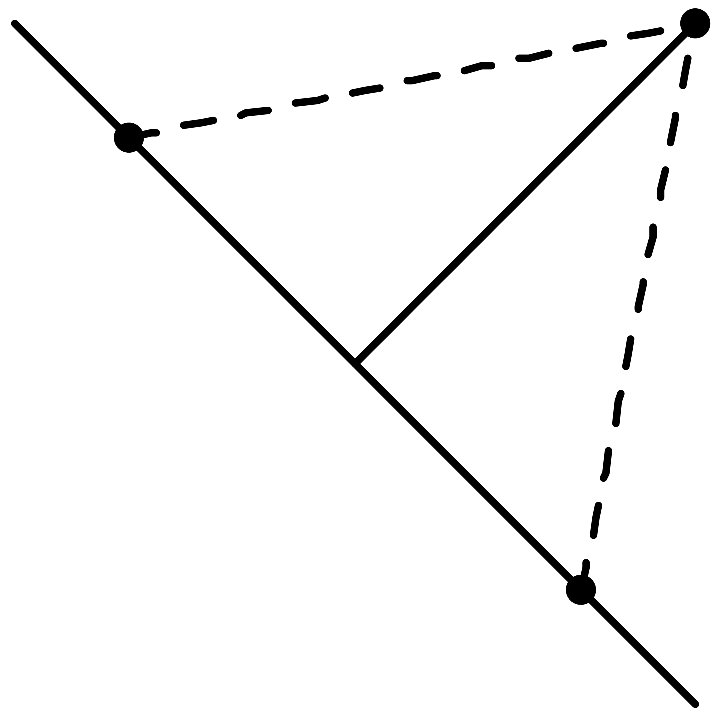
This agrees with the usual formulas for $\mathbf{R}^2$ and $\mathbf{R}^3$. Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

Then $\mathbf{u}-\mathbf{v} = (u_1 - v_1, u_2 - v_2)$ and

$$\operatorname{dist}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|(u_1 - v_1, u_2 - v_2)\|$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

Orthogonal Vectors



$$[\text{dist}(\mathbf{u},\mathbf{v})]^2 = \|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$$

$$= (\mathbf{u})\cdot(\mathbf{u}-\mathbf{v}) + (-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) =$$

$$= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + -\mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}$$

$$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v}$$

$$\Rightarrow \qquad [\text{dist}(\mathbf{u},\mathbf{v})]^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v}$$

Similarly,

$$[\text{dist}(\mathbf{u},-\mathbf{v})]^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}$$

Since $[\text{dist}(\mathbf{u}, -\mathbf{v})]^2 = [\text{dist}(\mathbf{u}, \mathbf{v})]^2$, $\mathbf{u} \cdot \mathbf{v} =$ **____**.

Two vectors **u** and **v** are said to be **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$.

Also note that if **u** and **v** are orthogonal, then $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

THEOREM 2 THE PYTHAGOREAN THEOREM

Two vectors **u** and **v** are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Section 6.2 Orthogonal Sets

A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in $\mathbf{R}^n$ is called an **orthogonal set** if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

EXAMPLE: Is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ an orthogonal set?

Solution: Label the vectors $\mathbf{u}_1, \mathbf{u}_2,$ and $\mathbf{u}_3$ respectively. Then

$\mathbf{u}_1 \cdot \mathbf{u}_2 =$

$\mathbf{u}_1 \cdot \mathbf{u}_3 =$

$\mathbf{u}_2 \cdot \mathbf{u}_3 =$

Therefore, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set.

THEOREM 4

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbf{R}^n$ and $W =$span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$. Then S is a linearly independent set and is therefore a basis for W.

Partial Proof: Suppose

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p = \mathbf{0}$$

$$(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p) \cdot \qquad\qquad = \mathbf{0} \cdot$$

$$(c_1\mathbf{u}_1) \cdot \mathbf{u}_1 + (c_2\mathbf{u}_2) \cdot \mathbf{u}_1 + \cdots + (c_p\mathbf{u}_p) \cdot \mathbf{u}_1 = \mathbf{0}$$

$$c_1(\mathbf{u}_1 \cdot \mathbf{u}_1) + c_2(\mathbf{u}_2 \cdot \mathbf{u}_1) + \cdots + c_p(\mathbf{u}_p \cdot \mathbf{u}_1) = \mathbf{0}$$

$$c_1(\mathbf{u}_1 \cdot \mathbf{u}_1) = \mathbf{0}$$

Since $\mathbf{u}_1 \neq \mathbf{0}$, $\mathbf{u}_1 \cdot \mathbf{u}_1 > 0$ which means $c_1 =$ ____.

In a similar manner, $c_2,\ldots,c_p$ can be shown to by all 0. So S is a linearly independent set.■

An **orthogonal basis** for a subspace W of $\mathbf{R}^n$ is a basis for W that is also an orthogonal set.

EXAMPLE: Express $\mathbf{y} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ as a linear combination of the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Solution:

$$\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1} = \qquad \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2} = \qquad \frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3} =$$

Hence

$$\mathbf{y} = \_\_\_\_\_\mathbf{u}_1 + \_\_\_\_\_\_\mathbf{u}_2 + \_\_\_\_\_\_\mathbf{u}_3$$

Orthonormal Sets

A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in $\mathbf{R}^n$ is called an **orthonormal set** if it is an orthogonal set of unit vectors.

If $W =$span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for W.

Recall that $\mathbf{v}$ is a unit vector if $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathbf{v}^T\mathbf{v}} = 1$.

Suppose $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$ where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

Then $U^TU = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$

$$= \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

It can be shown that $UU^T = I$ also. So $U^{-1} = U^T$ (such a matrix is called an **orthogonal matrix**).

EXAMPLE: Let $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}$.

Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Write **y** as the sum of a vector in W and a vector orthogonal to W.

Solution:

$$\text{proj}_W \mathbf{y} = \hat{\mathbf{y}} = \left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\right)\mathbf{u}_2$$

$$= \left(\quad\right)\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \left(\quad\right)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 9 \end{bmatrix}$$

EXAMPLE: Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for a subspace W of $\mathbf{R}^4$. Describe an orthogonal basis for W.

Solution: Let

$$\mathbf{v}_1 = \mathbf{x}_1 \text{ and } \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1.$$

$\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for Span$\{\mathbf{x}_1, \mathbf{x}_2\}$.

Let

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

(component of $\mathbf{x}_3$ orthogonal to Span$\{\mathbf{x}_1, \mathbf{x}_2\}$)

Note that $\mathbf{v}_3$ is in W. Why?

$\Rightarrow$ $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for W.

THEOREM 11 THE GRAM-SCHMIDT PROCESS

Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a subspace W of $\mathbf{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W and

$$\text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_p\} = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$$

EXAMPLE Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$, is a basis for a subspace W of $\mathbf{R}^4$. Describe an orthogonal basis for W.

Solution: $\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$ and

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{5}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{9}{14} \\ \frac{9}{7} \\ -\frac{15}{14} \\ 0 \end{bmatrix}$$

Replace $\mathbf{v}_2$ with $14\mathbf{v}_2$: $\mathbf{v}_2 = 14\begin{bmatrix} \frac{9}{14} \\ \frac{9}{7} \\ -\frac{15}{14} \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix}$

(optional step - to make $\mathbf{v}_2$ easier to work with in the next step)

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} - \frac{9}{630}\begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} - \frac{1}{70}\begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \\ 0 \\ 1 \end{bmatrix}$$

Rescale (optional): $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ 0 \\ 5 \end{bmatrix}$

Orthogonal Basis for W:

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 5 \end{bmatrix} \right\}$$

Orthonomal Basis

Suppose the following is an orthogonal basis for subspace

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}:$$

$$\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$$

Rescale to form unit vectors:

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \frac{1}{3} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Orthonormal basis for W : $\{\mathbf{u}_1, \mathbf{u}_2\}$