Examples

BS(Mathematics)-III

19 September 2012

A common puzzle

A child collected 50 spiders and beetles into a little box. When he counted the legs he found that there were altogether 340. How many beetles and how many spiders did he collect?

Example-1(Chemical equations)

Balance the following chemical equation

$$CH_4 + O_2 \to CO_2 + H_2O.$$

Balancing this equation means that there should be the same number of Carbon, Hydrogen and Oxygen atoms on both sides. Suppose

$$x_1 CH_4 + x_2 O_2 \to x_3 CO_2 + x_4 H_2O.$$

We need to find the numbers x_1, x_2, x_3 and x_4.

We have the following equations for Carbon, Hydrogen and Oxygen atoms respectively.

$$x_1 = x_3$$

$$4x_1 = 2x_4$$

$$2x_2 = 2x_3 + x_4$$

Solution

The matrix form of the equations is given by

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{pmatrix}$$

and the reduced echelon form of the matrix is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{pmatrix}$$

Hence we have the equations $x_1 - \frac{1}{2}x_4 = 0$, $x_2 - x_4 = 0$, $x_3 - \frac{1}{2}x_4 = 0$.

Solution Continued...

This means we have the equations

$$x_1 = \frac{1}{2}x_4, x_2 = x_4, x_3 = \frac{1}{2}x_4.$$

This clearly means we can choose the value of x_4 to get the value for the rest of the variables. So there are infinitely man possibilities to balance the chemical equation. But the main aim is use the fewest number of atoms. That will come from the smallest possible value of x_4. Hence the solution of the system is given by

$$x_1 = 1, x_2 = 2, x_3 = 1, x_4 = 2.$$

Circuit Laws

Ohm's Law

The voltage drop across a resistor is the product of the current passing through it and its resistance; that is,

$$E = IR.$$

Kirchhoff's Current Law

The sum of the currents flowing into any point equals the sum of the currents flowing out from the point.

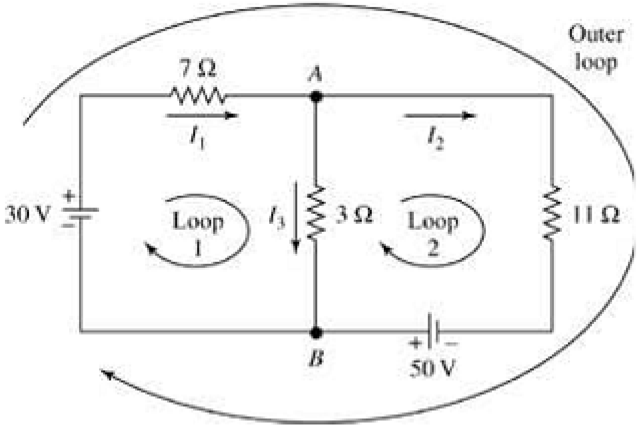
Kirchhoff's Voltage Law

Around any closed loop, the sum of the voltage rises is equal to to the sum of the voltage drops.

Conventions for Kirchhoff's Voltage Law

- A voltage drop occurs at a resistor if the direction assigned to the current through the resistor is the same as the direction assigned to the loop, and a voltage rise occurs at a resistor if the direction assigned to the current through the resistor is opposite to the direction assigned to the loop.
- A voltage drop occurs at a battery if the direction assigned to the loop is from $+$ to $-$ through the battery, and a voltage rise occurs at a battery if the direction assigned to the loop is from $-$ to $+$ through the battery.

Example-2 (Circuit Analysis)



Solution

Apply Kirchhoff 's current law to points A and B (respectively) to get

$$I_1 = I_2 + I_3$$

and

$$I_2 + I_3 = I_1$$

.

Applying Kirchhoff's voltage law and Ohm's law to loop 1 yields

$$7I_1 + 3I_3 - 30 = 0$$

similarly applying Kirchhoff's voltage law and Ohm's law to loop 2 yields

$$11I_2 - 3I_3 - 50 = 0.$$

What have we got here?

Some very simple examples on matrix multiplication

Q1: Can you multiply the following two matrices?

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 4 & -1 & 6 \end{pmatrix}$$

$$B = \begin{pmatrix} 7 & -2 & 4 \\ 4 & -6 & 3 \end{pmatrix}$$

The answer is NO! The matrices can only be multiplied if and only if

number of colums of A = number of rows of B.

Q2: Product of two matrices in different order!

$$A = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

Is AB and BA equal?
The answer again is NO.
Hence the matrix multiplication is not commutative in general.

Q3: Calculate AB and AC for the following matrices.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$$

Are they equal?
The answer is Yes. Is this true with numbers?

Final example

Q4: Calculate AB.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix}$$

Is $AB = 0$? If so what is you conclusion?

Inverse of a matrix

The inverse of a real number a is denoted by a^{-1}. For example $7^{-1} = \frac{1}{7}$.

Definition

An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the inverse of A .

Fact

Inverse of a matrix A is always unique. We will denote the inverse of A by A^{-1}.

Inverse of a 2×2 matrix

The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case the inverse if given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

.

Examples

Find the inverse of A and B.

$$A = \begin{pmatrix} 6 & 1 \\ 5 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}$$

Some questions regarding inverse

Q1: What is the inverse of AB?
Q2: What is the inverse of A^2?
Q3: What is the inverse of $5A$?

Thank You