Fourier Transform

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t}\, dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t}\, df$$

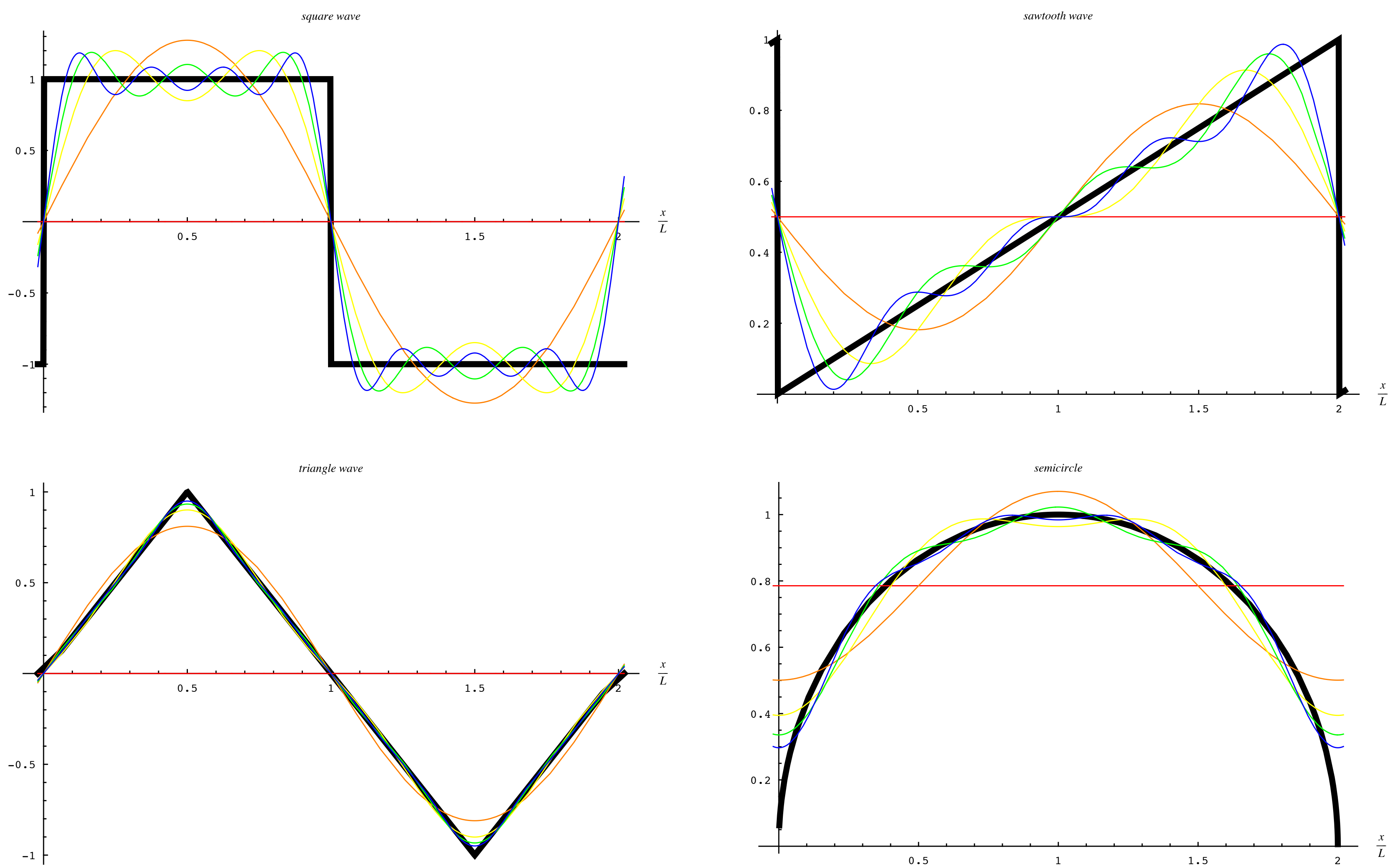
Fourier Series

- Fourier series are named in honor of Joseph Fourier (1768-1830).
- A Fourier series decomposes a periodic function or periodic signal into a sum of simple oscillating functions, namely sines and cosines (or complex exponentials).

Fourier Series

- A Fourier series is an expansion of a ***periodic*** function in terms of an ***infinite*** sum of sines and cosines.
- The computation and study of Fourier series is known as harmonic analysis and is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem, or an approximation to it, to whatever accuracy is desired or practical.

Fourier Series



Examples of successive approximations to common functions using Fourier series are illustrated above.

Fourier Series

Using the method for a generalized Fourier series, the usual Fourier series involving sines and cosines is obtained by taking $f_1(x)=\cos(x)$ and $f_2(x)=\sin(x)$. Since these functions form a complete orthogonal system over $[-\pi,\pi]$, the Fourier series of a function $f(x)$ is given by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(n x) + \sum_{n=1}^{\infty} b_n \sin(n x),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, d x$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(n x)\, d x$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(n x)\, d x$$

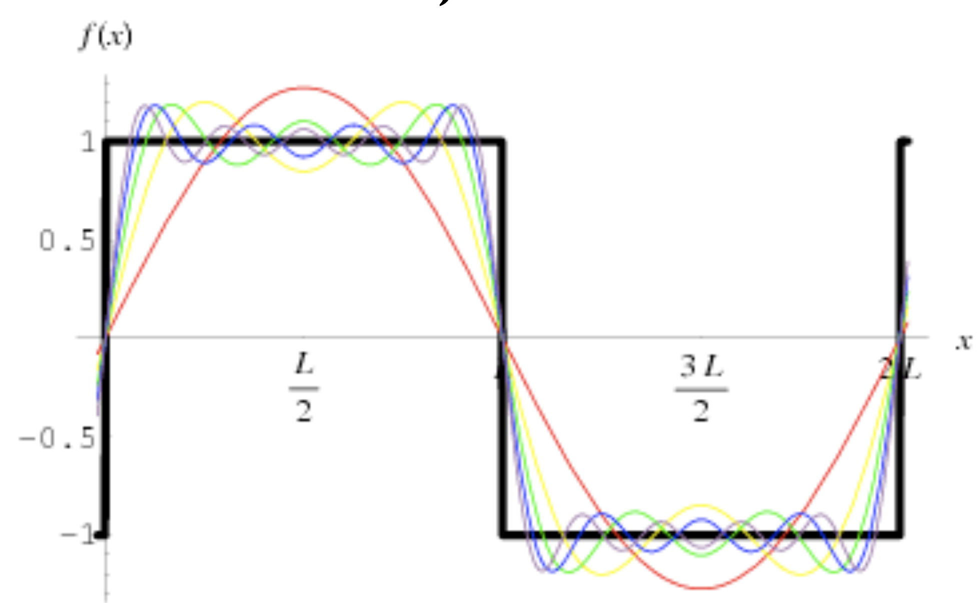
and $n=1, 2, 3, \ldots$.

Fourier Series

A Fourier series converges to the function $\overline{f}$ (equal to the original function at points of continuity or to the average of the two limits at points of discontinuity)

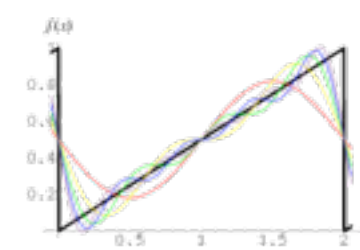
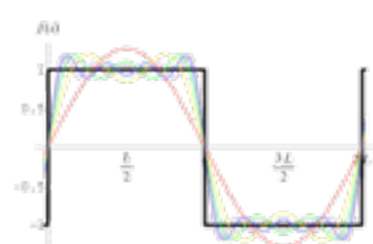
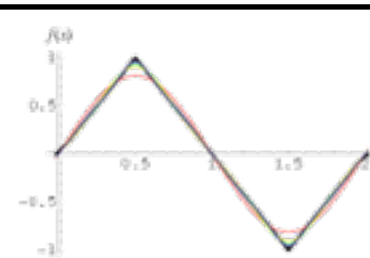
$$\overline{f} \equiv \begin{cases} \frac{1}{2}\left[\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right] & \text{for } -\pi < x_0 < \pi \\ \frac{1}{2}\left[\lim_{x\to -\pi^+} f(x) + \lim_{x\to \pi_-} f(x)\right] & \text{for } x_0 = -\pi, \pi \end{cases}$$

As a result, near points of discontinuity, a "ringing" known as the *Gibbs phenomenon*, can occur.

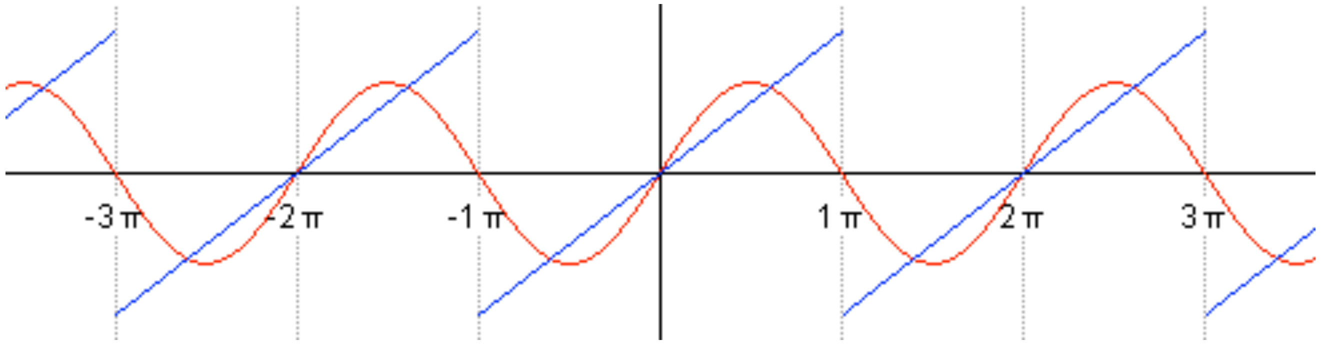


Fourier Series

The coefficients for Fourier series expansions of a few common functions are:

Function	$f(x)$	Fourier series	Plot
sawtooth wave	$\frac{x}{2L}$	$\frac{1}{2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)$	
square wave	$2\left[H\frac{x}{L}-H\left(\frac{x}{L}-1\right)\right]-1$	$\frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)$	
triangle wave	$T(x)$	$\frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin\left(\frac{n\pi x}{L}\right)$	

First five successive partial Fourier series



$$f(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)$$

Fourier Series

Fourier applied this technique to find the solution of the heat equation, publishing his initial results in 1807 and 1811, and publishing his Théorie analytique de la chaleur in 1822.

Fourier Series

The notion of a Fourier series can also be extended to complex coefficients.

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{inx}$$

Fourier Analysis

In Fourier analysis, the term Fourier transform often refers to the process that ***decomposes*** a given function into the basic ***pieces***. This process results in another function that describes how much of each basic piece are in the original function.

The original concept of Fourier analysis has been extended over time to apply to more and more abstract and general situations, and the general field is often known as ***harmonic analysis***.

Each transform used for analysis has a corresponding inverse transform that can be used for synthesis.

Fourier Transform

A physical process can be described either in the *time domain* as a function of time, e.g. $h(t)$, or in the frequency domain by giving the amplitude H as a function of frequency, $H(f)$. $H(f)$ is generally a complex number indicating phase also.

It is useful to think of $h(t)$ and $H(f)$ as being two different representations of the same function. We go back and forth between the two representations by means of the *Fourier transform equations*.

Fourier Transform Equations

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t}\, dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t}\, df$$

If *t* is measured in seconds, then *f* is in cycles per second (Hertz). If *h* is a function of position *x* in meters, *H* will be a function of inverse wavelength (cycles per meter).

Reading Assignment

Read Chapter 12 of Numerical Recipes

Class assignment

- Type `doc fft` and do the examples given
- In the help browser, select search results, type `sunspot` and open the first entry, do the example of time series analysis with the FFT that looks for periodicity in historical data on sunspot activity.
- In the help browser, select search results, type `animation` in the search field, select the first result and work through the examples, one of these is animating an FFT (`Visualizing an FFT as a Mov`).
- In the help browser click on the contents tab, open the `Matlab` section by clicking on the triangle next to the Matlab book, open the `Mathematics` section, then the `fourier transforms` section, then click on the `Fast Fourier Transform` section, work through the examples.
- Type `doc movie` at the command line, work through the examples.