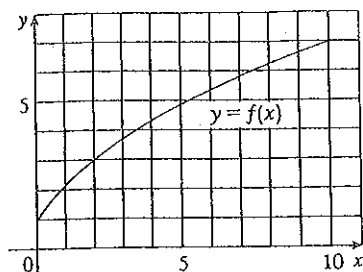1. (a) By reading values from the given graph of f, use five rectangles to find a lower estimate and an upper estimate for the area under the given graph of f from $x = 0$ to $x = 10$. In each case sketch the rectangles that you use.

   (b) Find new estimates using ten rectangles in each case.

2. (a) Use six rectangles to find estimates of each type for the area under the given graph of f from $x = 0$ to $x = 12$.
   - (i) L_6 (sample points are left endpoints)
   - (ii) R_6 (sample points are right endpoints)
   - (iii) M_6 (sample points are midpoints)

   (b) Is L_6 an underestimate or overestimate of the true area?

   (c) Is R_6 an underestimate or overestimate of the true area?

   (d) Which of the numbers L_6, R_6, or M_6 gives the best estimate? Explain.
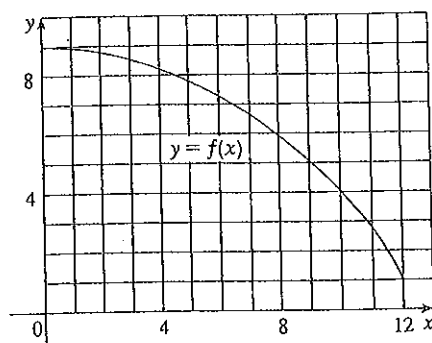
3. (a) Estimate the area under the graph of $f(x) = 1/x$ from $x = 1$ to $x = 5$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

   (b) Repeat part (a) using left endpoints.

4. (a) Estimate the area under the graph of $f(x) = 25 - x^2$ from $x = 0$ to $x = 5$ using five approximating rect-

   (b) Use a computer algebra system to find the sum in your expression from part (a).

   (c) Evaluate the limit in part (a).

[CAS] 22. Find the exact area of the region under the graph of $y = e^{-x}$ from 0 to 2 by using a computer algebra system to evaluate the sum and then the limit in Example 3(a). Compare your answer with the estimate obtained in Example 3(b).

[CAS] 23. Find the exact area under the cosine curve $y = \cos x$ from $x = 0$ to $x = b$, where $0 \le b \le \pi/2$. (Use a computer

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angles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

   (b) Repeat part (a) using left endpoints.

5. (a) Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.

   (b) Repeat part (a) using left endpoints.

   (c) Repeat part (a) using midpoints.

   (d) From your sketches in parts (a)–(c), which appears to be the best estimate?

6. (a) Graph the function $f(x) = e^{-x^2}$, $-2 \le x \le 2$.

   (b) Estimate the area under the graph of f using four approximating rectangles and taking the sample points to be (i) right endpoints and (ii) midpoints.
   In each case sketch the curve and the rectangles.

   (c) Improve your estimates in part (b) by using 8 rectangles.

7–8 ⋅ With a programmable calculator (or a computer), it is possible to evaluate the expressions for the sums of areas of approximating rectangles, even for large values of n, using looping. (On a TI use the Is> command or a For-EndFor loop, on a Casio use Isz, on an HP or in BASIC use a FOR-NEXT loop.) Compute the sum of the areas of approximating rectangles using equal subintervals and right endpoints for $n = 10$, 30, 50, and 100. Then guess the value of the exact area.

7. The region under $y = x^4$ from 0 to 1

8. The region under $y = \cos x$ from 0 to $\pi/2$

[CAS] 9. Some computer algebra systems have commands that will draw approximating rectangles and evaluate the sums of their areas, at least if x_i^* is a left or right endpoint. (For instance, in Maple use leftbox, rightbox, leftsum, and rightsum.)

   (a) If $f(x) = 1/(x^2 + 1)$, $0 \le x \le 1$, find the left and right sums for $n = 10, 30$, and 50.

   (b) Illustrate by graphing the rectangles in part (a).

   (c) Show that the exact area under f lies between 0.780 and 0.791.

[CAS] 10. (a) If $f(x) = \ln x$, $1 \le x \le 4$, use the commands discussed in Exercise 9 to find the left and right sums for $n = 10$, 30, and 50.

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algebra system both to evaluate the sum and compute the limit.) In particular, what is the area if $b = \pi/2$?

24. (a) Let A_n be the area of a polygon with n equal sides inscribed in a circle with radius r. By dividing the polygon into n congruent triangles with central angle $2\pi/n$, show that $A_n = \frac{1}{2}nr^2 \sin(2\pi/n)$.

   (b) Show that $\lim_{n \to \infty} A_n = \pi r^2$. [Hint: Use Equation 3.4.2 on page 214.]

(b) Illustrate by graphing the rectangles in part (a).

(c) Show that the exact area under f lies between 2.50 and 2.59.

11. The speed of a runner increased steadily during the first three seconds of a race. Her speed at half-second intervals is given in the table. Find lower and upper estimates for the distance that she traveled during these three seconds.

t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
v (ft/s)	0	6.2	10.8	14.9	18.1	19.4	20.2

12. Speedometer readings for a motorcycle at 12-second intervals are given in the table.

(a) Estimate the distance traveled by the motorcycle during this time period using the velocities at the beginning of the time intervals.

(b) Give another estimate using the velocities at the end of the time periods.

(c) Are your estimates in parts (a) and (b) upper and lower estimates? Explain.

| t (s)	0	12	24	36	48	60
v (ft/s)	30	28	25	22	24	27

13. Oil leaked from a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at two-hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.

| t (h)	0	2	4	6	8	10
$r(t)$ (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

14. When we estimate distances from velocity data, it is sometimes necessary to use times $t_0, t_1, t_2, t_3, \ldots$ that are not equally spaced. We can still estimate distances using the time periods $\Delta t_i = t_i - t_{i-1}$. For example, on May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table, provided by NASA, gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters. Use these data to estimate the height above Earth's surface of the space shuttle *Endeavour*, 62 seconds after liftoff.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

15. The velocity graph of a braking car is shown. Use it to estimate the distance traveled by the car while the brakes are applied.
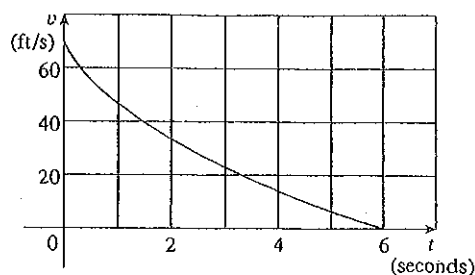
16. The velocity graph of a car accelerating from rest to a speed of 120 km/h over a period of 30 seconds is shown. Estimate the distance traveled during this period.
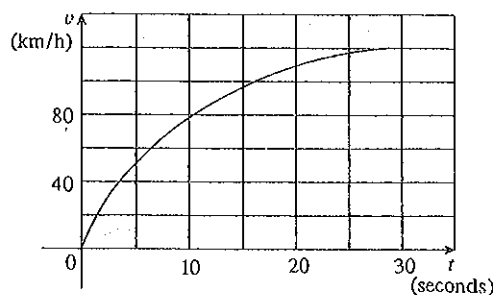
17–18 Use Definition 2 to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

17. $f(x) = \sqrt[4]{x}, \quad 1 \leqslant x \leqslant 16$

18. $f(x) = \dfrac{\ln x}{x}, \quad 3 \leqslant x \leqslant 10$

19. Determine a region whose area is equal to

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$$

Do not evaluate the limit.

20. (a) Use Definition 2 to find an expression for the area under the curve $y = x^3$ from 0 to 1 as a limit.

(b) The following formula for the sum of the cubes of the first n integers is proved in Appendix F. Use it to evaluate the limit in part (a).

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

[CAS] **21.** (a) Express the area under the curve $y = x^5$ from 0 to 2 as a limit.