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AP Physics C – Electromagnetism at a Glance

Much of physics is devoted to understanding the forces that push and pull the objects in our universe. Of the four fundamental forces—namely, the gravitational, electromagnetic, weak, and strong forces—arguably the one we best understand is the topic of this semester: electromagnetism. As the mass of a particle governs how strongly it is affected by the gravitational pull of a planet or any other source of gravitational field, an object's **electric charge** is a fundamental property of matter that determines how an object behaves in an electromagnetic field. The unit of charge in SI units is the **coulomb** (C).

As intrinsically interesting as physics is, we also study electromagnetism for our own selfish (i.e., practical) purposes. Since much of the matter in our world is charged, we can interact with it by generating electromagnetic fields in the way we want and, through some creativity and ingenuity, come up with wonders such as electricity, which has shaped our modern world more than any other technological innovation.

Today's lecture is a rough overview of the basic theory of classical electromagnetism. I can almost guarantee that you will not remember most of the specifics from today; rather, the purpose of this lecture is to give you a "heads up" of what you will encounter in the coming semester, so as to give you an idea of what to expect.

Electricity and magnetism: an overview

In general, there are two manifestations of the electromagnetic field:

- **Electric fields** affect charged objects;
- **Magnetic fields** affect *moving* charged objects.

We usually use $\vec{E}$ to represent the electric field and $\vec{B}$ to represent the magnetic field (both of which are vector quantities).[1]

In general, the the force on a charged particle of charge q and velocity $\vec{v}$ in an electromagnetic field is given by

$$\vec{F} = \underbrace{q\vec{E}}_{\text{electric force}} + \underbrace{q\vec{v} \times \vec{B}}_{\text{magnetic force}} . \tag{1}$$

[1] There exists another way to talk about the magnetic field, called the H-field, which comes up in the study of magnetic fields in materials, but we'll stick with B-field in this course.

Part of the total electromagnetic force comes from the electric force ($q\vec{E}$), and part comes from the magnetic force ($q\vec{v} \times \vec{B}$), also known as the *Lorentz force* when written in this form. As you can see, the electric force is directly proportional to the electric field $\vec{E}$; the constant of proportionality is the charge q.

Similarly, the magnetic field is directly proportional to the magnetic field $\vec{B}$. The constant of proportionality in this case is also q. However, the magnetic force additionally depends on the component of $\vec{v}$ perpendicular to $\vec{B}$ (this is what is meant by $\vec{v} \times \vec{B}$), means that the magnetic force only affects *moving* charged particles, as we claimed earlier.

The whole of electromagnetism is essentially defined by four equations, known as **Maxwell's equations**, that tell us how to determine $\vec{E}$ and $\vec{B}$. We could plop down Maxwell's equations right now, but you probably wouldn't understand what they mean. Even if you did, you wouldn't know how to apply them, just as knowing $\vec{F} = m\vec{a}$ alone does tell you much about how to do mechanics (e.g., you also need to develop the machinery of energy, momentum, and rotational motion).

The overall qualitative picture from Maxwell's equations is this:

- There two ways in which electric fields are produced:
  - By charged objects (Coulomb's law/Gauss's law) or
  - By a changing magnetic field (Faraday's law);
- And two ways in which magnetic fields are produced:[2]
  - By moving charged objects (Ampère's law/Biot-Savart law) or
  - By a changing electric field (Maxwell's correction to Ampère's law).

Of course, Maxwell's equations have much quantitative content as well. We will spend the rest of the lecture following the historical development of electromagnetism. Along the way, we will introduce the various laws (listed in parentheses above) that comprise Maxwell's equations. It is these laws that were eventually assembled into the form, namely Maxwell's equations, that we understand today.

[2]A quantum mechanical property called *spin* can also give rise to a magnetic field (which, at a fundamental level, is actually how magnets work), but we can ignore this for our purposes.

Electric fields I: Coulomb's law

The ancients discovered that rubbing certain objects together resulted in the two objects becoming charged and attracting each other. For instance, if we were to rub a piece of fur on a brass rod, the fur and rod would temporarily attract each other. The more one rubbed, the more charged the objects became and the more strongly they would attract. However, at times charged objects would repel each other: two rubbed brass rods repel, not attract. Through a trial-and-error process of repeated rubbings of different objects, it was determined that there were two varieties of charge, which we now designate positive and negative.[3] When two objects of like charge (i.e., both positive or both negative) were brought together, they would repel, while two objects of opposite charge (i.e., one positive and one negative) attracted each other.

As we stated earlier, in the SI system we measure charge in terms of coulombs (C). One coulomb is quite a "large" quantity of charge, as the charge of a single electron is only about 10^{-19} C; even trillions or hundreds of trillions of electrons have a total charge around the microcoulomb (10^{-6}C) range.

By the end of the eighteenth century, it was found that the force between two electric charges obeys an *inverse-square law*, just like the gravitational force; that is, the force between two charges is proportional to one over the square of the distance between them. Furthermore, the force is proportional to product of their charges. These two findings are the essence of **Coulomb's law**, which expresses the magnitude of the electric force F between two point charges in terms of the magnitudes of their charges q_1 and q_2 and the distance r between them:

$$F = k\frac{q_1 q_2}{r^2}. \tag{2}$$

The constant of proportionality is k, which we sometimes call the *Coulomb constant*, but which we usually write in terms of another constant ϵ_0, which we call the *electric constant* or *vacuum permittivity*:

$$k = \frac{1}{4\pi\epsilon_0}. \tag{3}$$

For us this fact seems a little arbitrary, but rest assured that the factor 4π helps simplify other laws in electricity and magnetism.[4]

[3] We could have called the two types of charge "Peter" and "Paul" or "A" and "B," but it is most mathematically convenient to designate charge by a sign.

[4] You may have noticed that Coulomb's law looks exactly like the law of gravitation,

The forces on the two charges are equal and opposite, which satisfies Newton's third law. The charges attract if they are of the same sign and repel if they are of opposite sign. For more than two charges, the *principle of superposition* applies; to calculate the force exerted on one charge by two other charges, we simply find the force provided by one charge and the force provided by the other change and add them together. Keep in mind, however, that force is a vector quantity (it has a direction), so we can't simply add the magnitudes of the two forces together.

> **Example 1** What is the magnitude of the electric force between a proton and an electron in a hydrogen atom? Compare this to the size of the gravitational force between them.
>
> Taking $r = 5.29 \cdot 10^{-11}$ m (the so-called *Bohr radius* for the hydrogen atom) and $q_1 = q_2 = 1.60 \cdot 10^{-19}$ C, Equation 2 gives $F_e = 8.24 \cdot 10^{-8}$ N. The force is attractive since the charge on the proton and electron are of opposite sign. While this force may seem small at first glance, it is much, much larger than the gravitational force at the same distance, which we find to be on the order of 10^{-47} N. (The mass of a proton is $1.67 \cdot 10^{-27}$ kg; the mass of an electron is about a thousand times smaller, $9.11 \cdot 10^{-31}$ kg.) Thus, on the atomic (but not nuclear) scale, the electric force is by far the most important force. □

There is nothing wrong with talking about electric *force*, but we will find that it is more convenient to express things in terms of something called field (to which we have alluded thoughout but not described). We say that the magnitude of the **electric field** E at a point located a distance r away from a charge of magnitude Q is equal to

$$E = k\frac{Q}{r^2} \tag{4}$$

and that the force $\vec{F}$ on a charge q where the electric field is $\vec{E}$ is

$$\vec{F} = q\vec{E}. \tag{5}$$

As you can see, in terms of describing the force from electric charges, there is really no difference between using the Coulomb's law for an electric

except with the gravitational constant G switched with k and mass m_1 and m_2 replaced by charge q_1 and q_2, respectively. Except for the fact that there are two types of charge, electrostatics is a carbon copy of gravitation.

field (Equations 4 and 5) and using Coulomb's law for electric force (Equation 2); in doing the former, we simply split Equation 2 into two separate equations. We like to use field instead of force for the simple reason that we might want to talk about some charge distribution without reference to the charge of the particle that feels the force. In other words, we don't necessarily know beforehand the charge of the hypothetical particle for which we might want to calculate electric force, in which case we don't want to make reference to a hypothetical number. It would be a little awkward to say, for example, that the force on a particle at a certain point would be such and such *given* that its charge is such and such. Instead, if we want to talk about some distribution of charges, we can simply give the field at some point, without reference to the charge that we would place in the field. Later, if we want to actually know the force for a certain particle, we simply multiply the field by the particle's charge.

Electric fields II: Gauss's law

To compute the field produced by a collection of point charges, such as the positive and negative ions in a salt crystal, Coulomb's law serves us well. However, if we wish to deal with *continuous* distributions of charge, such as a charged plate or sphere, Coulomb's law is a little unruly to use—imagine adding up the forces from infinitely many charges, which point in all different directions. Sometimes this is necessary (which we can do with calculus), but luckily we can re-express Coulomb's law in an equivalent form called **Gauss's law**, which is particularly useful when dealing with charge distributions in which there is some degree of symmetry.

Before we state Gauss's law, let us introduce a new quantity called *flux*. Given an arbitrary closed, three-dimensional surface of our choosing (a so-called *Gaussian surface*), the **electric flux** at each point on the surface is, roughly speaking, the size of the perpendicular component of the field that at each point, multiplied by the infinitesimal surface area surrounding that point. To find the flux over the entire surface, we sum over all the points on the surface. The general mathematical expression for flux involves vector calculus.

It suffices for our purposes to know that if the flux at each point is constant over the entire surface (that is, if the component of the electric field perpendicular to the surface is constant throughout the surface), the total electric flux ϕ (the Greek letter *phi*) over the surface is given by the simple relationship

$$|\phi| = EA, \tag{6}$$

where E is the magnitude of the electric field and A is the area of the surface. ϕ is positive if the electric field points out of the surface and negative if it points into the surface.

Example 2 Find the electric flux over a spherical surface (of radius r) that contains a charge of magnitude q at its center.

The flux is constant over the entire surface, so we are free to use Equation 6. By Coulomb's law, the magnitude of the electric field is

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

everywhere on the sphere. The area of the sphere is $A = 4\pi R^2$, so the electric flux is

$$\phi = EA = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}(4\pi r^2) = \frac{q}{\epsilon_0}.$$

ϕ has the correct sign: if q is positive, the electric field points outward, and ϕ is positive; if q is negative, the electric field points inward, and ϕ is negative. Finally, note that the flux is independent of r. □

Gauss's law says that the electric flux over a Gaussian surface is proportional to the total charge enclosed within that surface. The constant of proportionality is ϵ_0. Where the electric flux is ϕ, we have

$$\phi = \frac{Q}{\epsilon_0}, \tag{7}$$

where Q is the quantity of charge enclosed within the Gaussian surface. Proving Gauss's law requires vector calculus, so we won't go into that here, but rest assured that Gauss's law works and is equivalent to Coulomb's law.

Example 3 Show that Gauss's law is equivalent to Coulomb's law for a single point charge by (a) using Coulomb's law to show that $\phi = Q/\epsilon_0$ and (b) independently using Gauss's law to calculate E (and showing that this agrees with Coulomb's law).

We accomplished part (a) in the previous example. To do part (b), we again use a spherical Gaussian surface of radius r with a point charge (of charge Q) at the center. Thus

$$E = \frac{\phi}{A} = \frac{Q}{\epsilon_0 A}.$$

$A = 4\pi r^2$, from which it follows that

$$E = \frac{Q}{4\pi\epsilon_0 r^2}. \qquad \square$$

Using Gauss's law, it is a simple task to show that the electric field inside a thin, hollow charged spherical orb of uniform surface charge density (charge per unit area) is zero.[5] The reasoning is as follows. If we take as our Gaussian surface a sphere inside and concentric to the orb, there is no charge enclosed, so the total electric flux must be zero. By symmetry, the flux along the surface is constant, so $EA = 0$, which implies that $E = 0$. We can do this for all concentric spheres smaller than the orb, so the electric field is zero at any point inside the orb.

We can apply the same reasoning to points outside of the orb by examining spheres larger than the orb. In particular, we use as our Gaussian surface a sphere of radius r. Here, the charge enclosed is the total charge Q on the sphere. Again, we reason that the electric field everywhere along our surface is the same by symmetry, in which case the magnitude of the field E must be the same everywhere on the surface. Thus the electric flux is $EA = 4\pi r^2 E$ and Gauss's law (Equation 7) gives that

$$4\pi r^2 E = \frac{Q}{\epsilon_0},$$

so

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}. \tag{8}$$

This is the same as the field produced by a point charge of charge Q located a distance r away. (In fact, you can probably start to see why we made k more complicated in Coulomb's law; Gauss's law, which is actually more fundamental in a sense, became simpler.) Thus, as far as the electric field outside a uniform spherical orb is concerned, an orb is no different from a point charge.

Via a similar process, using cylindrical Gaussian surfaces we can show that the field produced by an infinite sheet of uniform charge density σ (again, recall this is the charge per area) is $\sigma/(2\epsilon_0)$, an important result that

[5] We could also use Coulomb's law, but the process is much more strenuous. If you've studied gravity before in a previous physics class, you may have seen an analogous derivation done for a hollow spherical shell of uniform mass density—the process is the same since the law of gravitation and Coulomb's law are both inverse-square laws.

we will call upon later in the lecture. The field points perpendicular from the sheet, away for a positively charged sheet and toward for a negatively charged sheet. We can do the same thing with charged cylindrical shells and so forth. To find the field of solid objects, all we have to do is add up lots of thin shells or sheets.

> **Example 4** Determine the electric field between two oppositely charged parallel sheets of the same uniform charge density σ. Assume that the plates are placed parallel to the ground and that the bottom plate is positively charged. What is the field outside the plates?
>
> In the region between the plates, the field produced by the positively charged (bottom) plate points upward, with magnitude $\sigma/(2\epsilon_0)$. The field produced by the negatively charged (top) plate points upward as well, also with magnitude $\sigma/(2\epsilon_0)$. Since the fields point in the same direction, by superposition the total field in this region points upward and is of magnitude σ/ϵ_0. In the region below the plates, the field produced by the positively charged plate points downward, while the field produced by the negatively charged plate points upward, so the fields cancel. Similarly, in the region above the plates, the field produced by the positively charged plate points upward, while the field produced by the negatively charged plate points downward, and the fields cancel. Therefore, the electric field outside the parallel plates is zero. □

Before we move on to magnetism, let us discuss one more situation that frequently arises in electricity. Suppose we have an ideal conductor—that is, a material, such as metal, in which charges are essentially free to move without resistance—and free charge scattered in that conductor. We might want to know what happens to those charges and determine the charge distribution in the conductor after the charges come to rest in some equilibrium position.

There are some relatively complicated setups possible, but a couple simple facts quickly point us in the right direction. First, when the charges come to rest, the electric field inside the conductor must be zero; otherwise, there would be a net force on a charge inside the conductor, which would mean that the charges have not come to rest. Second, at the edge of the conductor, the insulating material or vacuum surrounding the conductor prevents charge from hopping off of the surface, so the electric field can be nonzero at the surface. However, the electric field at the surface must be perpendicular to the surface, or else charge would continue moving sideways along the surface.

It turns out that there exists a unique charge distribution for a given setup that satisfies the two conditions (E must be perpendicular to the surface and zero inside). Sometimes this distribution is difficult to find, but for several setups that possess some degree of symmetry, there are a few techniques that we can use to find a charge distribution that creates only electric fields perpendicular to the surface of the conductor. Note that the conditions hold even if there exist some external charges. In the presence of an external electric field, the charges in the conductor merely reposition themselves such that the two conditions are satisfied.[6] In particular, this concept lends itself to an important application in *electromagnetic shielding.* If we wish to protect some sensitive electronics or helpless human from some unwanted electric field, all we have to do is enclose the objects to be protected inside a closed conductor, in which case the electric field inside should be zero.

Magnetic forces and fields

Just as the earliest civilizations vaguely knew about the existence of electric forces, they also knew a bit about magnetic forces, the apparent source of which were naturally occurring magnetic rocks or the earth. The study of the two seemingly unrelated forces diverged down separate paths for centuries until 1820, when Hans Christian Orsted noticed that an electric current, which is essentially moving charge, attracts a magnet. In other words, moving charge generates a magnetic field. From that moment on, scientists realized that electricity and magnetism were related; later it was also discovered that magnetic fields exert a force on moving charges in the form of a current-carrying wire.

First, we examine the effect of magnetic fields on moving charges.[7] Recall that the *Lorentz force* describes the effect of a magnetic field $\vec{B}$ on a single point charge q moving at a velocity $\vec{v}$:

$$\vec{F}_B = q\vec{v} \times \vec{B}, \tag{9}$$

[6]Note that neutral charge in a conductor is simply the superposition of positive and negative charge. Although the total charge into a neutral conductor is zero, if some of the charges move, regions of the conductor may not be electrically neutral.

[7]It might seem most natural to start with a discussion of the force exerted on a magnet by another magnet and perhaps come up with an "analogue" of Coulomb's law for magnetic forces, but somewhat ironically this is not easy. There do exist relatively "simple" expressions for the force between two magnets, but deriving them is not a simple task, as magnets are not as fundamental as, say, moving charges are.

where the magnetic field $\vec{B}$ is measured in units of teslas (T), which is equivalent to $\text{N}/(\text{C} \cdot \text{m/s})$.

The *cross product* ($\times$) means that the force $\vec{F}_B$ acts in a direction mutually perpendicular to $\vec{v}$ and $\vec{B}$, in accordance with a mathematical rule called the *right-hand rule*: point your right thumb in the direction of $\vec{v}$, your pointer in the direction of $\vec{B}$, and your middle finger will give the direction of $\vec{F}_B$. Second, the size of $\vec{F}_B$ depends not only on the size of the charge, the charge's velocity, the strength of the magnetic field, but also the relative direction of $\vec{v}$ and $\vec{B}$, since it turns out mathematically that

$$|\vec{v} \times \vec{B}| = |\vec{v}||\vec{B}| \sin \theta, \tag{10}$$

where θ is the (acute) angle between $\vec{v}$ and $\vec{B}$. Thus, the more "perpendicular" that $\vec{v}$ and $\vec{B}$ are to each other (the greater $\sin \theta$ is), the more force is exerted on the particle.

Example 5 A magnetic field of constant magnitude B points outward from the plane of the page. A particle of mass m enters with some initial speed v in the plane of the page . Explain why the particle undergoes centripetal motion in the plane, and determine the radius of its path.

The magnetic field is always perpendicular to the plane, so the right-hand rule gives a magnetic force that always lies in the plane. Furthermore, this magnetic force is always perpendicular to $\vec{v}$ by default, so it generates centripetal motion in the plane. Where R is the radius of the path, the magnitude of the centripetal force equals mv^2/R, in which case

$$qvB = \frac{mv^2}{R},$$

which gives

$$R = \frac{mv}{qB}.$$

Centripetal motion resulting from a magnetic field can be observed in many real-life situations, one of which has to do with the *solar wind*, essentially just charged particles ejected from the sun. As the particles approach the earth's magnetic field, they move in helices (why?) around the earth's magnetic field lines. □

We can use Equation 9 to determine the force exerted on a straight wire by a magnetic field $\vec{B}$. As we will discuss more in depth in our next lecture, the current I of a given wire (is basically defined as the rate at which charge flows through the wire, which we assume to be the same at all points in the wire. For a short segment of the wire dl, the amount of time it takes for a charge moving at a speed v to travel that length is dl/v, in which time a total charge $I(dl)/v$ has passed through that segment. Multiplying by $\vec{v} \times \vec{B}$ then gives $Id\vec{l} \times \vec{B}$, where $\vec{l}$ is the direction of the wire, which is the same thing as the direction of the velocity. Thus for the entire wire we have

$$\vec{F} = I\vec{l} \times \vec{B}. \tag{11}$$

Now we examine how moving charges *generate* magnetic fields. Experimentally, we find that the magnetic field generated *by* a moving particle at a given point in space is proportional to the particle's speed, charge, and square of the distance between the point and particle. The magnetic field also depends on the angle between the the line connecting the particle and point and the velocity of the particle: the closer to perpendicular, the greater the field. Mathematically, we have

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2}, \tag{12}$$

where q and $\vec{v}$ are the charge and velocity, respectively, of the particle; $\hat{r}$ gives the direction (but not magnitude[8]) of the line connecting the charge and the given point; and r^2 gives the square of the distance between the charge and given point. μ_0 is known as the *magnetic constant* or *vacuum permeability* and, along with the electric constant ϵ_0, is a fundamental constant.

Example 6 Two identical positively charged particles, each of charge q and constant speed v, travel antiparallel to each other (i.e., they move in opposite directions but on paths parallel to each other). At the moment at which the two particles are closest together, find the electric and magnetic forces exerted on one of the particles.

The magnitude of the magnetic field B felt by each of the particles is given by

$$B = \frac{\mu_0}{4\pi} \frac{qv}{r^2}.$$

The cross product disappears since the velocity of each particle is perpendicular to the line connecting the two particles. Thus the magnitude of the magnetic force F_B exerted on each particle is

$$F_B = qvB = \frac{\mu_0}{4\pi} \frac{q^2 v^2}{r^2}.$$

[8] $\hat{r} = \vec{r}/|\vec{r}|$ is a unit vector, so it has magnitude equal to one.

The direction of this force, meanwhile, is away from the other particle. To find the magnitude of the electric force F_E, we simply use Coulomb's law, which yields

$$F_E = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}. \qquad \square$$

As Equation 12 contains a cross product, we can already start to see that working out the direction of things might get complicated, especially given a continuum of charge in the form of current. Using the same reasoning we used to find Equation 11 from the Lorentz equation (Equation 9), we can find the an equivalent expression for Equation 12 in terms of I, $\vec{l}$, and and $\vec{r}$; the resulting formula is known as the **Biot-Savart law**. We will elucidate its use in a future lecture.

Without proof we will give a simple but important result. The magnitude of the magnetic field B generated by a straight wire carrying a current I is

$$B = \frac{\mu_0 I}{2\pi r} \tag{13}$$

for some point a distance r from the wire. Due to the cross product in Equation 12, the field points in a clockwise direction if we were to sight down the wire in the direction of the current. One way to find this visually is to point your thumb in the direction of the current and curl your fingers around the wire, a mnemonic known as the *right-hand rule* (in addition to the right-hand rule we learned for cross products). The direction your fingers point is the direction of the magnetic field.

Example 7 Two parallel wires located a distance r apart each carry a current I (both in the same direction). Find the force per unit length exerted on one wire by the other.

Where F is the force exerted on some arbitrary length l of wire, Equations 11 and 13 give

$$F = IlB = Il\frac{\mu_0 I}{2\pi r},$$

so

$$\frac{F}{l} = \frac{\mu_0 I^2}{2\pi r}. \qquad \square$$

So far we have examined electric fields resulting from the presence of charged particles. But it also turns out, as we mentioned at the very beginning of the lecture, that changing magnetic fields alone can create an electric field. By 1831, Michael Faraday had made the surprising discovery that a changing magnetic field resulted in an electric field, a phenomenon called *electromagnetic induction*. Its implications are many and great. One of these is the fact that we can generate electricity by moving a magnet in the right way, which we can do by expending some energy source (e.g., coal or gas) to power a turbine.

In general, the size of the *electric potential* (we'll define exactly what this term means when we get to our lecture on electric circuits) generated in a planar loop of wire by a changing magnetic field is equal to the rate of change of *magnetic flux*, which is similar to electric flux. Magnetic flux is the size of the perpendicular component of the field that at each point on some surface, times the surface area surrounding each point, summed over our entire surface. In this case, we take the surface to be the area enclosed by the loop. There are many ways to change magnetic flux; for example, we can move the loop farther away from the source of magnetic field (thus effectively decreasing the magnetic field) or rotate the loop (thus changing the component of the field perpendicular to the loop). One way of stating **Faraday's law** is as follows:

$$\mathcal{E} = -\frac{d\phi}{dt}, \tag{14}$$

where $\mathcal{E}$ is the electric potential across the loop and ϕ is the magnetic flux across the loop.

Maxwell's equations and electromagnetic waves

Given the fast pace of discovery throughout the first half of the 19th century, the high-water mark of classical electricity and magnetism came in 1865 at the hands of James Clerk Maxwell. In his treatise *A Dynamical Theory of the Electromagnetic Field*, Maxwell summarized all the results of electricity and magnetism in four succinct equations, which we reproduce below for the sake of completeness:[9]

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{15}$$

$$\nabla \cdot \vec{B} = 0 \tag{16}$$

[9]This is the modern *differential form* of Maxwell's equations in SI units. Elsewhere you may instead find the so-called *integral form* of Maxwell's equations, as well as another set of units for describing electricity and magnetism, namely *Gaussian units*, which are preferred by many physicists.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{17}$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J} \tag{18}$$

Everything probably looks like gibberish unless you know some multivariable calculus (and like random mathematics unless you've had some more physics), but that doesn't stop us from explaining what each of these laws mean intuitively. The *nabla sign* (∇) is the so-called *del operator*, which is used to express operations in vector calculus. Thus, the first two equations concern some operations on electric fields $\vec{E}$, and the last two concern magnetic fields $\vec{B}$. $\vec{J}$ represents the *current density*, which is really nothing more than the current per area.

Their order doesn't really matter, but traditionally the equations are listed in the order we have given.

- The first of the four equations is Gauss's law (Equation 15), which we've already examined briefly.
- The second (Equation 16) is the statement that there exist no *magnetic monopoles*; all magnets have both a south and a north end.[10]
- The third equation (Equation 17) summarizes Faraday's law of electromagnetic induction. You'll notice that the time-derivative of the magnetic field (the rate of change of magnetic field) appears in the equation.
- Finally, the fourth equation (Equation 18) is a statement of the Biot-Savart law for current-carrying wires and is called **Ampère's law**.
  - The first term says that a magnetic field results from current (density) $\vec{J}$.
  - The other term, $\mu_0 \epsilon_0 \partial \vec{E}/\partial t$, says that a magnetic field results from a changing electric field. This term, which Maxwell added for reasons we won't go into, was a purely theoretical discovery that he made.

But Maxwell did a lot more than simply summarize results, or today we wouldn't extol him today as the greatest scientist between Newton and Einstein. Maxwell's genius lies in the fact that he collected and tidied up

[10]Find a magnetic monopole and within a few years you will have one of the surest Nobel Prizes in your hands.

all everything that was known about electricity and magnetism as one unified field and examined its implications. The addition of the extra term $\mu_0\epsilon_0\partial\vec{E}/\partial t$, known as *displacement current* (without the μ_0), was one of his many fragments of insight.

Additionally, in *A Dynamical Theory*, Maxwell also showed that it was possible to construct a wave of propagating electric and magnetic fields that satisfy both Maxwell's equations and the classical wave equation, which describes continuous waves such as sound. In other words, there could exist *electromagnetic waves* through space. Amazingly, these waves, Maxwell demonstrated, must move with speed c, where

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}}. \tag{19}$$

The use of the letter c here is no accident. It so happened, to the surprise of many, that $1/\sqrt{\epsilon_0\mu_0}$ was none other than the speed of light, approximately 3×10^8 m/s in a vacuum. In fact, this agreement between the speed of an electromagnetic wave and the speed of light was so coincidental that few doubted that there could only be one conclusion: light is just an electromagnetic wave! This is another deep fact that we won't have time to really talk about, but note a couple things. Since the electric and magnetic fields have direction, it turns out that individual light waves are *polarized*; they have a particular orientation. (Modern polarized sunglasses take advantage of this principle by allowing only light polarized in certain directions to pass through and not others.) More importantly, we can generate electromagnetic waves from changing electric and magnetic fields.

Relativity and electromagnetism

There was one last problem, which wasn't an issue with Maxwell's equations per se, but an issue with classical physics. What happens to Maxwell's laws in a moving frame? It turns out that if we want Maxwell's equations to hold in a moving frame in accordance with classical physics, lots of stuff goes wrong. First of all, notice that the magnetic force $q\vec{v} \times \vec{B}$ on a charge depends on the velocity $\vec{v}$ of the charge. What's wrong with this? Well, all inertial (constant-velocity) frames should be equivalent. However, in different frames (for instance, a frame moving in the same direction as but slower than a charged particle), the magnetic force on a moving charge is different; in other words, $\vec{v}$ is frame dependent, so the magnetic force is also frame dependent. In particular, in the frame of the charge, the magnetic force disappears! Certainly, the laws of physics do not appear to be the same in all inertial frames.

Either Maxwell was wrong or classical physics was wrong. Instead of doubting the centuries-old foundation of Newton, the scientists of Maxwell's time instead chose to doubt Maxwell's equations, which were relatively new. Later, Einstein's theory of *special relativity* finally resolved the issue for one and for all, instead finding fault with classical physics and not Maxwell's equations. It so happens that we can begin to derive the phenomenon of magnetism by operating in the frame of a moving charge and transforming the resulting electric force from frame to frame in accordance with the principles of relativity. In this way, electricity and magnetism were finally unified as the two manifestations of the same force, *electromagnetism*.