1. We consider the following balls and bins problem.

   We have M balls labeled $1, 2, 3, \ldots, M$ and two bins A and B. At each time slot, we uniformly pick a random number $k \in \{1, 2, 3, \ldots, M\}$ and we remove ball k from the bin where it is and put it back in either A or B, chosen uniformly at random.

   Let X_n denote the number of balls in bin A at time slot n.

   (a) Let $M = 4$.

      i. Prove that $(X_n)_{n \geq 0}$ is a Markov chain.

      ii. Describe its transition matrix and draw its transition diagram. Justify your answer.

      iii. Is the chain irreducible? Is it aperiodic?

      iv. Derive its stationary distribution π.

      v. In this question, we assume that the number of balls is M, some positive integer, instead of 4. Describe the content of bin A after running the dynamics described above for a long time. Justify your answer.

   (b) In what follows, we suppose that $M = 4$

      i. We suppose that the bin A is initially empty, i.e. $X_0 = 0$ What is the probability that it contains an even number of balls after we run the above dynamics for a long time? Justify your answer.

      ii. We suppose that the bin A initially contains all four balls, i.e. $X_0 = 4$. We observe the evolution of the above Markov chain, i.e. $X_0, X_1, X_2, \ldots$. How often do we observe more balls in A than in B.

2. We study a model of the dynamics of the spread of an infectious disease in a population of healthy people of size N.

   We assume that there are X_n infected individuals and $S_n = N - X_n$ healthy ones by day n. Between day n and $n+1$, each of the S_n healthy individuals has a probability p of meeting a given infected individual and thus contracting the disease. Moreover, in day $n+1$ all the X_n infected individuals, in day n, recover and become healthy.

   (a) We assume that there are 3 individuals with $p = 1/3$.

      i. Describe the transition matrix of the Markov Chain X_n.

      ii. Classify the states of the chain according to whether they are recurrent or transient.

      iii. Does X_n has a stationary distribution? Comment.

      iv. Compute the average number of days before the epidemic stops starting with two infected individuals.

   (b) We now assume that the population size is N and that the contact probaility $p \in (0, 1)$.

      i. Justify the fact that if there are i infected individuals then the probability that a given healthy individual becomes infected in the following day is given by

$$1 - (1-p)^i$$

ii. Derive an expression for the transition probabilities

$$p_{ij} = P(X_{n+1} = j | X_n = i)$$

3. (a) Let us assume that $(X_t)_{t \geq 0}$ is a given continuous-time Markov chain on the state space S with rate matrix $Q = (q_{ij})_{i,j \in S}$. Show that if there exists a probability distribution $\pi = (\pi_i)_{i \in S}$ such that, for all $i, j \in S$, we have

$$\pi_i q_{ij} = \pi_j q_{ji} ,$$

then π is the invariant distribution of $(X_t)_{t \geq 0}$. The Markov chain is said to be *reversible*.

Hint: Use the fact that if Q is a rate matrix then $\sum_{j \in S} q_{ij} = 0$, for all $i \in S$.

(b) We consider the dynamics of a Markovian single server queue in continuous time. Customers join the queue and are served on a first-in-first-out basis, i.e., according to the order in which they join the queue. We suppose that the time between two successive arrivals is exponentially distributed with parameter $\lambda > 0$ and that each customer requires a service time that is exponentially distributed with parameter $\mu > 0$.

Let X_t, $t \geq 0$, be the random process that describes the number of customers waiting in the queue including the one being served.

i. Derive the transition matrix Q of $(X_t)_{t \geq 0}$.

ii. Using question 3. a), show that, if $\rho = \frac{\lambda}{\mu} < 1$, the stationary (invariant) distribution of $(X_t)_{t \geq 0}$ is given by

$$\pi_i = (1 - \rho)\rho^i .$$

Comment on the *stability condition* $\rho < 1$.

iii. Compute the average number of customers in the queue in the stationary regime.

(c) Let us now assume that we have a post office with two cashiers. The two queues at the cashiers run (separately) in parallel. Each of these queues operates following the dynamics in 3) b) with the same parameters λ and μ.

i. Derive the average number of customers in the post office.

ii. A clever employee suggests merging the two queues so that customers arrive at rate 2λ wait until one of the two cashiers is available and is then served. Describe the underlying continuous-time Markov chain and discuss its stability.

iii. Using similar arguments as in 3. b), show that the average number of customers in the post office in stationary regime is given by

$$\frac{2\mu\lambda}{(\mu + \lambda)(\mu - \lambda)} .$$

iv. Is the suggestion of the employee better than running the two queues separately? Justify your answer.