Chapter 3 Polynomials

As described in the introduction of Chapter 1, applications of solving linear equations arise in a number of different settings. In particular, we will in this chapter focus on the problem of modelling a continuous real function f on some interval $[a, b]$. We will begin by illustrating this in the context of *polynomial interpolation* where we will construct polynomials that exactly match the function f at certain fixed points of the interval $[a, b]$. We will then investigate an other alternative to interpolation that is more suited to approximate polynomials. To this we will define the continuous analogue of the discrete linear least-squares studied in Chapter 2.

1 Polynomial interpolation

The problem of interpolating a function $f : [a, b] \to \mathbb{R}$ can be stated as: Given f continuous on the interval $[a, b]$ and $n + 1$ points $\{x_0, x_1, \dots, x_n\}$ satisfying $a \le x_0 < x_1 < \cdots < x_n \le b$, determine a polynomial $P_n \in \mathbb{R}_n[X]$ such that

$$P_n(x_i) = f(x_i), \quad \forall i = 0, \dots, n ,$$

where $P_n \in \mathbb{R}_n[X]$ means that

$$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i$$

and we say that the polynomial P_n has degree n.

We require $n + 1$ data points to construct an interpolating polynomial of degree n since if the number of points is smaller than n, then we could construct infinitely many degree n interpolating polynomials and if it is larger, then there would be no degree n interpolant.

Monomial basis The most straightforward way of solving the interpolation problem is to notice that for the choice of the interpolant $P_n(x) = a_0 + a_1 x + \dots a_n x^n = \sum_{i=0}^{n} a_i x^i$ we have

$$\begin{aligned} a_0 + a_1 x_1 + \cdots + a_n x_1^n &= f(x_1) \\ a_0 + a_1 x_2 + \cdots + a_n x_2^n &= f(x_2) \\ &\vdots \\ a_0 + a_1 x_n + \cdots + a_n x_n^n &= f(x_n) . \end{aligned}$$

In matrix form this can be rewritten as

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix} .$$

Matrices of this form are called Vardemonde matrices and they are invertible since their determinant is given by $\prod_{i=0}^{n} \prod_{j=i+1}^{n} (x_j - x_i)$ which is non-zero if the x_is are distinct. In particular, this implies that there is a unique interpolant to f satisfying the above conditions. To solve the linear system above, we require $O(n^3)$

Despite being straightforward, the use of the monomial basis 1, x, x^2, ... for interpolating gives rise to unpleasant numerical problems when attempting to determine the coefficients a_i on a computer. The main issue stems for the fact that the monomial basis functions look increasingly similar as we take higher and higher powers. This in turn might lead to the coefficients c_i to become very large in magnitude even if the function f remains of modest size on $[a, b]$. To understand this phenomena, we can use an analogy with linear algebra. In particular, try to express the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in the basis $\begin{pmatrix} 1 \\ 10^{-10} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Lagrange basis An alternative to the monomial basis is to express the polynomial P_n in a different basis $L_0, L_1, \dots, L_n$ of polynomials of degree at most n in $\mathbb{R}_n[X]$, where $L_i(x_k) = 0$ for all $k \neq i$ and $L_i(x_i) = 1$. If such basis exists then it is not difficult to see that if we want to express $P_n(x)$, the interpolant of f in $x_0, \dots, x_n$, as a linear combination of the L_is then we have that

$$P_n(x) = \sum_{i=0}^{n} a_i L_i(x)$$

and $a_i = P_n(x_i)$. It is not difficult to check that the *Lagrange polynomials* defined by

$$L_i(x) = \frac{\prod_{k=0,k\neq i}^{n}(x - x_k)}{\prod_{k=0,k\neq i}^{n}(x_i - x_k)} ,$$

does the trick. With the Lagrange basis, finding the coefficients for the interpolation does not require to solve a linear equation.

Newton basis The monomial and the Lagrange bases represent the two extreme cases from a numerical perspective to perform interpolation. Moreover, both share the shortcoming that one needs to have all the data points and the values of f in advance to be able to perform the interpolation. In practise, these data points may arrive one after the other requiring the recomputation of the interpolant when one getsa new data point. It would therefore be useful if one could derive the interpolants recursively as the points are made available to us. In what follows we will construct a basis $Q_0, Q_1, \dots Q_n$ such that $P_n(x) = \sum_{i=0}^{n} a_i Q_i(x)$ and $P_k(x) = \sum_{i=0}^{k} a_i Q_i(x)$ interpolates f in the points $\{x_0, \dots, x_k\}$ for $k = 0, \dots, n$.

Let us start by constructing the polynomial P_0 of degree 0 that interpolates$f$ in x_0 clearly $P_0(x) = f(x_0)$. Let $P_0 = a_0 Q_0$ where $Q_0(x) = 1$ for all x and $a_0 = f(x_0)$. Now, we use P_0 to find P_1 and a Q_1 in $\mathbb{R}_1[X]$ that interpolates F in x_0 and x_1, i.e.

$$P_1(x) = P_0(x) + a_1 Q_1(x) .$$

Note that

$$P_1(x_0) = P_0(x_0) + a_1 Q_1(x_0) = f(x_0) + a_1 Q_1(x_0) .$$

As we require that $P_1(x_0) = f(x_0)$, this implies that $a_1 Q_1(x) = 0$, i.e. either $a_1 = 0$ (this only happens when $f(x_0) = f(x_1)$ while we are seeking a generic basis that works for all f) of $Q_1(x_0) = 0$. The latter condition yields

$$Q_1(x) = x - x_0$$

and

$$P_1(x) = a_0 + a_1(x - x_0)$$

To determine a_1, we use the interpolation condition for x_1 which implies that

$$a_1 = \frac{f(x_1) - a_0}{x_1 - x_0} .$$

Next, we determine P_2 in $\mathbb{R}_2[X]$ that interpolates f at x_0, x_1, x_2 where $P_2(x) = P_1(x) + a_2 Q_2(x)$ and Q_2 in $\mathbb{R}_2[X]$ to be determined. The interpolation conditions together with the properties of the polynomial P_1 imply that

$$Q_2(x_0) = Q_2(x_1) = 0 \quad \Rightarrow Q_2(x) = (x - x_0)(x - x_1) ,$$

and from $Q_2(x_2) = 0$ we get that

$$a_2 = \frac{f(x_2) - P_1(x_2)}{Q_2(x_2)} = \frac{f(x_2) - a_a - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} .$$

Following the same pattern, it is not difficult to see that one can construct $Q_1, Q_2, Q_3, \dots, Q_n$ a basis of $\mathbb{R}_n[X]$ such that, for $k = 0, \dots n$

$$Q_k(x) = \prod_{i=0}^{k-1} (x - x_i)$$

where $P_k(x)$ the interpolant of f at $x_0, \dots, x_k$ is given by $P_k(x) = \sum_{i=0}^{k} a_i Q_i(x)$ and

$$a_i = \frac{f(x_i) - \sum_{j=0}^{i-1} a_j Q_j(x_i)}{Q_i(x_i)} \, .$$

The above basis is known as the Newton basis. The entire procedure of constructing P_n can be written in matrix form as follows

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & (x_1 - x_0) & 0 & \dots & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 1 & (x_n - x_0) & (x_n - x_0)(x_n - x_1) & \dots & \prod_{i=0}^{n-1}(x_n - x_i) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix} .$$

With the newton form of the interpolant, one can easily update P_n to P_{n+1} to incorporate a new data point $(x_{n+1}, f(x_{n+1}))$. Note that solving the linear system to find the coefficients a_i takes $O(n^2)$ steps.

We would like emphasise that the three methods above lead the same polynomial P_n. It is just expressed in three different bases.

2 Orthogonal polynomials

Interpolation with high degree polynomials is not always the best way to approximate a function as the polynomial might diverge form the function is is meant to approximate at points in the intervals (x_i, x_{i+1}) (between the points of interpolation) as the degree of the polynomials increase. In fact the interpolation only accounts for the points of interest $x_0, x_1, \dots$ and ignores all other points. In this section, we consider an alternative to polynomial interpolation, namely polynomial approximation where we want to find a polynomial $P_n \in \mathbb{R}_n[X]$ such that $P_n(x_i) \approx f(x_i)$ for $i = 1, \dots, m$ where $m \geq n$. When $m = n$ we have seen that such a polynomial exists and is unique. when $m > n$ we must settle with an approximation, together with a method for quantifying the error of this approximation. For example, we could choose to minimise the maximum error

$$\min_{P \in \mathbb{R}_n[X]} \max_{0 \leq i \leq m} |f(x_i) - P(x_i)|$$

or the sum of the squares of the errors

$$\min_{P \in \mathbb{R}_n[X]} \sum_{i=0}^{m} (f(x_i) - P(x_i))^2 \, .$$

Alternatively, one could ignore the specific points and measure the entire interval of interest $[a, b]$

$$\min_{P \in \mathbb{R}_n[X]} \max_{x \in [a,b]} |f(x) - P(x)| \quad \text{or} \quad \min_{P \in \mathbb{R}_n[X]} \int_a^b (f(x) - P(x))^2 dx \, .$$

Polynomial approximation In this section we will focus the latter error that can be seen as the continuous analogue of the standard linear least-squares method.

Example 1 *Find a polynomial $P \in \mathbb{R}_1[X]$ that minimises $\int_0^1 (e^x - P(x))^2 dx$.*

In general suppose that we are interested in expressing the polynomial P_n that minimises $\int_a^b (f(x)-P(x))^2 dx$ in a bais $\phi_1, \ldots, \phi_n$ of $\mathbb{R}_n[X]$, i.e.

$$P(x) = \sum_{i=0}^{n} a_k \phi_k(x) \ .$$

In this case

$$E(a_0, \ldots, a_n) = \int_a^b (f(x) - P(x))^2 dx = \langle f, f\rangle - 2\sum_{i=0}^{n} a_i \langle f, \phi_i\rangle + \sum_{i=0}^{n}\sum_{k=0}^{n} a_i a_k \langle \phi_i, \phi_k\rangle$$

where $\langle g, h\rangle = \langle h, g\rangle = \int_a^b g(x)h(x)dx$. In particular to minimise $E(a_0, \ldots, a_n)$ we need to find a_i such that $\partial E/\partial x_i = 0$ which implies

$$\langle f, \phi_i\rangle = \sum_{k=0}^{n} a_k \langle \phi_i, \phi_k\rangle$$

In matrix form this can be rewritten as

$$\begin{pmatrix} \langle \phi_0, \phi_0\rangle & \langle \phi_0, \phi_1\rangle & \ldots & \langle \phi_0, \phi_n\rangle \\ \langle \phi_1, \phi_0\rangle & \langle \phi_1, \phi_1\rangle & \ldots & \langle \phi_1, \phi_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \phi_n, \phi_0\rangle & \langle \phi_n, \phi_1\rangle & \ldots & \langle \phi_n, \phi_n\rangle \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \langle f, \phi_0\rangle \\ \langle f, \phi_1\rangle \\ \vdots \\ \langle f, \phi_n\rangle \end{pmatrix} .$$

Suppose $a = 0$, $b = 1$ and $\phi_k(x) = x^k$ then

$$\langle \phi_i, \phi_j\rangle = \int_0^1 x^{i+j} dx = \frac{1}{i+j+1} \ .$$

The corresponding matrix is the well-known Hilbert matrix. This is a poor basis (supporting our earlier findings when using the monomials for interpolating). In fact Hilbert matrices have very large condition numbers, e.g. $\kappa(H) \approx 10^{14}$ where H is the Hilbert matrix of order 10 and it increases the larger the matrix.

Orthogonal polynomials and continuous least squares Here we will focus on more general errors by investigating polynomials that minimise $\int_a^b (f(x) - P(x))^2 w(x) dx$ for w a continuous, positive function on $[a, b]$ and we let

$$\langle g, h\rangle = \langle h, g\rangle = \int_a^b g(x)h(x)w(x)dx \ .$$

Definition 1 *We will say that two functions g and h are* orthogonal *if $\langle g, h\rangle = 0$. Moreover, a set of function $\phi_0, \phi_1, \ldots, \phi_n$ is a system of orthogonal polynomials if*

- *for all k ϕ_k is a polynomial of exact degree k*
- *for all $k \neq j$, $\langle \phi_j, \phi_k\rangle = 0$.*

In particular if $\phi_0, \phi_1, \ldots, \phi_n$ are orthogonal polynomials then $(\phi_0, \phi_1, \ldots, \phi_n)$ is basis of $\mathbb{R}_n[X]$. Moreover if $P \in \text{Span}(\phi_0, \phi_1, \ldots, \phi_{n-1})$ then

$$\langle P, \phi_n\rangle = 0 \ .$$

We now describe a mechanism for constructing orthogonal polynomials. The Gram-Schmidt process used to orthogonalise vectors in $\mathbb{R}^n$ can in fact be employed here as well. More precisely, suppose that we have a basis of $\mathbb{R}_n[X]$, $P_0, P_1, \ldots, P_n$ where the degree of P_k is exactly k, as matter of example this could be the monomial basis $P_i(x) = x^i$ then the Gram-Schmidt algorithm takes the following form

Pseudocode: Gram-Schmidt for polynomials

1. Let $\phi_0 = P_0$
2. For $k = 1, \ldots, n$

   $\phi_k = p_k - \sum_{i=0}^{k-1} \frac{\langle p_k, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle} \phi_i$

Now suppose that one has a set of orthogonal polynomials $\phi_0, \ldots \phi_n$ and seeks the next orthogonal polynomial ϕ_{n+1}. Since the degree of ϕ_n is exactly n, then the degree of $x\phi_n(x)$ is $n+1$ and one could perform the Gram-Schmidt procedure on $\phi_0(x), \ldots \phi_n(x), x\phi_n(x)$ which forms a basis of $\mathbb{R}_{n+1}[X]$. In particular

$$\phi_{n+1}(x) = x\phi_n(x) - \sum_{i=}^{n} \frac{\langle x\phi_n(x), \phi_i(x) \rangle}{\langle \phi_i, \phi_i \rangle} \phi_i(x) \, .$$

First notice that

$$\begin{aligned}
\langle x\phi_n(x), \phi_i(x) \rangle &= \int_a^b x\phi_n(x)\phi_i(x)w(x)dx \\
&= \int_a^b \phi_n(x)x\phi_i(x)w(x)dx \\
&= \langle \phi_n(x), x\phi_i(x) \rangle
\end{aligned}$$

Since $x\phi_i(x)\mathbb{R}_{i+1}[X]$, then for $i < n-1$ we have $\langle \phi_n(x), x\phi_i(x) \rangle = 0$. this yields substantial simplification in the Gram-Schmidt procedure. In particular

Theorem 1 *Given a positive, continuous weight function w on $[a, b]$ and an associated inner product*

$$\langle g, h \rangle = \langle h, g \rangle = \int_a^b g(x)h(x)w(x)dx \, ,$$

then a system of orthogonal polynomials can be generated as follows

$$\begin{aligned}
\phi_0(x) &= 1 \\
\phi_1(x) &= x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \\
\phi_k(x) &= x\phi_{k-1}(x) - \frac{\langle x\phi_{k-1}(x), \phi_{k-1}(x) \rangle}{\langle \phi_{k-1}(x), \phi_{k-1}(x) \rangle} \phi_{k-1}(x) - \frac{\langle x\phi_{k-1}(x), \phi_{k-2}(x) \rangle}{\langle \phi_{k-2}(x), \phi_{k-2}(x) \rangle} \phi_{k-2}(x) \, .
\end{aligned}$$

Example 2 *On $[-1, 1]$ with wight function $w(x) = 1$ the orthogonal polynomials are knowns as* Legendre polynomials*:*

$$\begin{aligned}
\phi_0(x) &= 1 \\
\phi_1(x) &= x \\
\phi_2(x) &= x^2 - \frac{1}{3} \\
\phi_3(x) &= x^3 - \frac{3}{5}x \, .
\end{aligned}$$

A general expression for these polynomials is as follows

$$\phi_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \, ,$$

where $\frac{d^n}{dx^n}(x^2 - 1)^n$ stands for the n-th derivative of $(x^2 - 1)^n$.

Example 3 *If we change the interval* $[a, b]$ *and the weight we have a number of interesting families of orthogonal polynomials.*

- $[a, b] = [-1, 1]$, *and* $w(x) = \frac{1}{\sqrt{1-x^2}}$ *we have the notorious Chebyshev polynomials* T_n *that satisfy*

$$T_n(\cos(\theta)) = \cos(n\theta)\ .$$

- $[a, b] = (-\infty, \infty)$, *and* $w(x) = e^{-x^2}$ *we have the notorious Hermite polynomials* H_n *where*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}\ .$$

Let us go back to find the best approximation to a function f, given by

$$\begin{pmatrix} \langle \phi_0, \phi_0 \rangle & \langle \phi_0, \phi_1 \rangle & \dots & \langle \phi_0, \phi_n \rangle \\ \langle \phi_1, \phi_0 \rangle & \langle \phi_1, \phi_1 \rangle & \dots & \langle \phi_1, \phi_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \phi_n, \phi_0 \rangle & \langle \phi_n, \phi_1 \rangle & \dots & \langle \phi_n, \phi_n \rangle \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \langle f, \phi_0 \rangle \\ \langle f, \phi_1 \rangle \\ \vdots \\ \langle f, \phi_n \rangle \end{pmatrix}.$$

In $\phi_0, \dots, \phi_n$ is an orthogonal basis of $\mathbb{R}_n[X]$ then one can create an orthonormal basis $\psi_0, \dots, \psi_n$ where

$$\psi_k = \frac{\phi_k}{\langle \phi_k, \phi_k \rangle^{1/2}}\ .$$

and the coefficients a_k such that $P_n(x) = \sum_{k=0}^n a_k \psi_k(x)$ where P_n given the minimum of $\langle f(x) - P(x), f(x) - P(x) \rangle$ are then given by $a_k = \langle f, \psi_k \rangle$.

Theorem 2 *The unique* L^2 *approximation to* f*, i.e the one that minimises the distance* $||P - f||_{L^2}^2 = \int_a^b (f(x) - P(x)^2 w(x) dx$ *is given by*

$$P^* = \sum_{k=0}^n \langle f, \psi_k \rangle \psi_k\ ,$$

where $\langle f, \psi_k \rangle = \int_a^b f(x) \psi_k(x) w(x) dx$.

To prove this note that $\langle f - P^*, \psi_k \rangle = 0$ for all $k = 0, \dots, n$ and by consequence for all $Q \in \mathbb{R}_n[X]$ which can be rewritten as a linear combination of the ψ_ks we have $\langle f - P^*, Q \rangle = 0$

Example 4 *Approximate* $f(x) = e^x$ *for* $[a, b] = [0, 1]$ *and* $w(x) = 1$.

3 Exercises

Exercise 1 Consider the following inner product on $\mathbb{R}[X]$:

$$\langle P, Q\rangle = \int_{-1}^{1} \frac{P(x)\,Q(x)}{\sqrt{1-x^2}}dx \ .$$

1. Show that this is in fact an inner product.
2. For every $n \in \mathbb{N}$ there exists a unique polynomial T_n such that

   $$T_n(\cos x) = \cos(nx) \text{ for all } x \in \mathbb{R} \ .$$

   Verify this for $n = 1, \ldots, 4$.
   Write then a recursive representation for the general case (any n).
3. Verify: $\left(1-X^2\right) \cdot T_n''(X) - X \cdot T_n'(X) = -n^2 \cdot T_n(X)$ by using $X = \cos(x)$
4. Consider now $P = a_0 + a_1 x^1 + \ldots + a_n x^n \in \mathbb{R}_n[X]$. We represent it as $P = (a_0, a_1, \cdots a_n)^T$. Given this expression, find a matrix M such that $P \mapsto \left(1-x^2\right)P'' - xP'$. What are its range and kernel for $n = 3$?
5. Determine the eigenvalues amd eigenvectors of M.
6. Show the orthogonality of the polynomials.
   Hint: Use $x = \cos(\sigma)$ and substitute $T_n(x) = \cos(n\sigma)$

Exercise 2 (2011) We consider the set $\mathbb{R}_n[X]$ of polynomials with real coefficients and degrees less or equal to n endowed with the inner product $\langle P, Q\rangle = \int_{-1}^{1} P(t)Q(t)dt$.

1. Show that $\langle P, Q\rangle = \int_{-1}^{1} P(t)Q(t)dt$ is indeed an inner product on $\mathbb{R}_n[X]$.
2. Give the expression of $\langle P, Q\rangle$ when P and Q are polynomials in $\mathbb{R}_2[X]$ in terms of the coefficients of both P and Q.
3. Let L be the application on $\mathbb{R}_n[X]$ such that

   $$L(P) = \frac{d}{dX}\left[(X^2-1)\frac{dP}{dX}\right] \ .$$

   (a) Show that if $P \in \mathbb{R}_n[X]$ then $L(P) \in \mathbb{R}_n[X]$ and that L is a linear transformation on $\mathbb{R}_n[X]$.
   (b) Prove that, for all P, Q in $\mathbb{R}_n[X]$, we have

   $$\langle L(P), Q\rangle = \langle P, L(Q)\rangle \ .$$

   Hint: Perform integrations by parts.
4. Let $P_0 = 1$ and for $k = 1, \cdots, n$, define the polynomial P_k of degree k as follows

   $$P_k = \frac{d^k}{dX^k}\left((X^2-1)^k\right) \ ,$$

   the k-th derivative of $(X^2-1)^k$.

   (a) Compute P_1 and P_2.

(b) Derive an expression of $L(P_k)$ in terms of P_k' and P_k'' the first and second derivatives of P_k, respectively.

(c) Prove the following identity

$$(X^2-1)\frac{d\big[(X^2-1)^k\big]}{dX} - 2kX(X^2-1)^k = 0\,.$$

(d) By differentiating $(k+1)$ times the above expression, establish that

$$(X^2-1)P_k''(X) + 2XP_k'(X) = k(k+1)P_k(X)\,.$$

Hint: Use Leibniz's formula

$$(fg)^{(k+1)} = \sum_{i=1}^{k+1}\binom{k+1}{i} f^{(i)} g^{(k+1-i)}\,,$$

where $f^{(i)}$ is the i-th derivative of f.

(e) Find the eigenvalues and eigenvectors of the transformation L.

5. Let k, l two integers between 0 and n.

   (a) Express $\langle L(P_k), P_l\rangle$ and $\langle L(P_l), P_k\rangle$ in terms of $\langle P_k, P_l\rangle$.

   (b) Prove that $(P_0, P_1, \cdots, P_n)$ is an orthogonal basis of $\mathbb{R}_n[X]$ when endowed with the inner product $\int_{-1}^{1} P(t)Q(t)dt$.

These polynomials are known as Legendre polynomials.