Notes on the Vickrey Auction
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The Vickrey auction generalizes the second price auction to environments where more than one good is being sold. The goods can be identical, as in the Treasury Bill case, different but in a structured way where everyone agrees on the relative values, as in the Google case, or entirely different, as in the Spectrum case.

The model

- Seller has a set of goods $1, ..., K$ to sell.
- Bidders care about profit equal to value minus payment.
- Bidder i has value $v_i(x)$, defined for any subset of the goods x.

Vickrey rules

1. Bidders are asked to submit their (full list of) values.
2. Seller uses the submitted values to find the allocation that leads to the most total surplus, where total surplus is measured as the sum of the bidder values.
3. Seller computes payments. To compute i's payment p_i
   - Let v_i denote the value that i gets from the efficient allocation (if i is assigned a subset of goods x, then $v_i = v_i(x)$).
   - Let V denote the total surplus from the efficient allocation ($V = \sum_i v_i$)
   - Let V^{-i} denote the total surplus that could be generated if i did not participate (and the seller allocated the goods among the other bidders to maximize total value).
   - Let $p_i = v_i - (V - V^{-i})$ be i's Vickrey payment.

Comments

1. Assuming the bidders truthfully reveal their values, the profit that each bidder makes in the auction is exactly equal to the amount by which they increase social surplus. That is, bidder i's profit is $v_i - p_i = V - V^{-i}$. If i wasn't around the efficient allocation would create a surplus V^{-i}. With i around, the efficient allocation creates surplus V. Bidder i gets to keep this as profit.

2. A bidder who wins no items also pays nothing. To see this, notice that if the seller assigns the goods efficiently without i being around and when i shows up, it isn't efficient to give her anything, then $V = V^{-i}$, and also $v_i = 0$. So $p_i = 0$.

3. We can also interpret p_i as having i pay the seller the value she "displaces" by showing up at the auction and getting her efficient bundle of goods (the interpretation in class). To see why, notice that $V = \sum_j v_j$. So

$$p_i = v_i - (V - V^{-i}) = v_i - \sum_j v_j + V^{-i} = V^{-i} - \sum_{j \neq i} v_j.$$

   Now V^{-i} is the total value from assigning the goods efficiently to all bidders that aren't i, and $\sum_{j \neq i} v_j$ is the total value these bidders get when i shows up and is assigned some of the goods. So therefore $V^{-i} - \sum_{j \neq i} v_j$ is the value that the other bidders lose when i shows up and "displaces" them, and this is also p_i.

4. The optimal strategy for a bidder in a Vickrey auction is to reveal her true value. There is a proof of this in the Milgrom book, and the logic is similar to the second price auction although writing it out involves more notation.

Example: Single Object Vickrey Auction

In the second price auction, bidders are asked to submit their values. The high value wins the auction. If the high value is v_1, the total surplus created is $V = v_1$. Suppose the next highest value submitted is v_2. If the high value guy weren't around, the object would go to bidder 2, so $V^{-1} = v_2$. Using the Vickrey formula, therefore, the winner must pay $p_1 = v_1 - (V - V^{-1}) = v_2$, that is, the second highest stated value. So the single object Vickrey auction is just a second price auction!

To see a numerical example, suppose the submitted values are 10 and 8. Then the high value is 10, $V = 10$ and without the 10 bidder around, the maximum value would be 8, so $V^{-1} = 8$. Using the Vickrey formula, the high value guy pays 8.

Example: Multi-Unit Auction (T-Bills)

Suppose there are two identical goods. Suppose bidder 1 has value 120, bidder 2 has value 110 and bidder 3 has value 100. In the Vickrey auction, they submit their values, the seller gives 1 and 2 each a single unit, and they both pay 100. Why? Think about bidder 1. He has $v_1 = 120$. With him around $V = 230$. Without him around, $V^{-1} = 210$. So $p_1 = 100$. For bidder 2, $v_2 = 110$, $V = 230$ and $V^{-2} = 220$. So $p_2 = 100$. If everyone demands one unit, the Vickrey auction and the Treasury auction are the same.

Now imagine in the example that Bidder 2 wants two units. The submitted values are 120, (110, 110), and 100. The seller gives one unit to bidder 1 and one unit to bidder 2. She charges Bidder 1 a price of 110, and bidder 2 a price of 100! Why? In the case of bidder 1, $v_1 = 120$, $V = 230$, and if bidder 1 wasn't there the seller would give both units to bidder 2 so $V^{-1} = 220$. Thefore $p_1 = 110$. In the case of bidder 2, $v_2 = 110$, $V = 230$ and $V^{-2} = 220$. So $p_2 = 100$.

What if bidder 2 wanted only one unit but bidder 1 wanted two units? Then bidder 1 would get both units and would pay $p_1 = 210$. Try working it out on your own!

Example: Sponsored Search

Suppose there are two position, that get 200 and 100 clicks. There are three bidders with click values 3,2,1. The seller asks for the values, assigns the slots efficiently. The total surplus generated is $V = 3 \cdot 200 + 2 \cdot 100 = 800$. Bidder 1 gets value $v_1 = 3 \cdot 200 = 600$, and bidder 2 gets $v_2 = 200$. If bidder 1 weren't around, the seller would put bidder 2 in the top slot and then bidder 3, so $V^{-1} = 2 \cdot 200 + 1 \cdot 100 = 500$. If bidder 2 weren't around, then $V^{-2} = 700$. So using the Vickrey formula, the seller charges Bidder 2 $p_2 = 100$, or \$1 per click, and charges bidder 1 $p_1 = 300$, or \$1.50 per click.

Example: Spectrum Licenses

Consider the example from class with NY and SF for sale and three bidders who each want at most one license.

	NY	SF
A	40	35
B	60	50
C	80	60

In the Vickrey auction, the bidders submit their true values. The seller gives C the NY license and B the SF license. The total value created is $V = 80 + 50 = 130$. Bidder C gets $v_C = 80$, and B gets $v_B = 50$. If C weren't around, the A would get SF and B would get NY, so $V^{-C} = 35 + 60 = 95$. Therefore by the Vickrey formula, C must pay $p_C = v_C - \left(V - V^{-C}\right) = 80 - (130 - 95) = 45$ for the NY license. If B weren't around, A would get SF and C would get NY, so $V^{-B} = 80+35 = 105$. Therefore $p_B = v_B - \left(V - V^{-B}\right) = 50 - (130 - 105) = 35$ is what B pays for SF.