MAE 290A - Homework # 5
Numerical Methods in Science and Engineering

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Due date: Thursday, Nov 29, 2012

Problem 1 - Lagrange interpolation. Write a computer program for Lagrange interpolation. Using this program, we will perform interpolation on Runge's function, $y = (1 + 25x^2)^{-1}$ using the data in the table below. We wish to fit a smooth curve through the data using the Lagrange polynomial interpolation, for which the value at any point x is simply

$$P(x) = \sum_{j=0}^{n} y_j \prod_{i=0, i \neq j}^{n} \frac{x - x_i}{x_j - x_i}$$

x_i	-1.0	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8	1.0
y_i	0.038	0.058	0.1	0.2	0.5	1.0	0.5	0.2	0.1	0.058	0.038

(a) Test your program by verifying that $P(0.7) = -0.226$.

(b) Using the data above, find the interpolated value at $x = 0.9$. Plot the Lagrange polynomial in the interval between -1.0 and 1.0.

(c) Use Runge's function to generate a table of 21 equally spaced data points. Interpolate these data using a Lagrange polynomial of order 20. Plot this polynomial and comment on the comparison between this result and the plot in part a.

Problem 2 - Cubic Spline. The tuition for nine units at Whatsammatta U has been increasing from 1990 to 2008 as shown in the table below:

Year	Tuition per year
1990	\$3,516
1992	\$4,541
1994	\$5,483
1996	\$6,550
1998	\$7,825
2000	\$9,354
2002	\$10,816
2004	\$12,180
2006	\$13,509
2008	\$14,911

(a) Plot the given data points and intuitively (draw) a smooth curve through them.

(b) Interpolate the data with the Lagrange polynomial. Plot the polynomial and the data points. Use the polynomial to predict the tuition in 2011. This is an extrapolation problem. Discuss the utility of Lagrange polynomials for extrapolation.

(c) Repeat part (b) with a cubic spline interpolation and compare your results.

Problem 3 - Tension splines. Tension splines can be used if the interpolating polynomial wiggles too much. In this case, the equation governing the position of the plastic ruler (i.e. beam bending) in between the data points is

$$y'''' - \sigma^2 y'' = 0$$

where σ is the tension parameter. If we denote $g_i(x)$ as the interpolating tension spline in the interval $x_i \leq x \leq x_{i+1}$ then $g_i''(x) - \sigma^2 g_i(x)$ is a straight line in this interval, which can be written in the following form:

$$g_i''(x) - \sigma^2 g_i(x) = \left[g''(x_i) - \sigma^2 f(x_i)\right] \frac{x - x_{i+1}}{x_i - x_{i+1}} + \left[g''(x_{i+1} - \sigma^2 f(x_{i+1})\right] \frac{x - x_i}{x_{i+1} - x_i}$$

(a) Verify that for $\sigma = 0$, the cubic spline is recovered, and $\sigma \to \infty$ leads to linear interpolation.

(b) Derive the equation for the tension spline interpolation, i.e. the expression for $g_i(x)$.

Problem 4 - Piecewise Lagrange Interpolation. Consider a piecewise Lagrange polynomial that interpolates between three points at a time. Let a typical set of consecutive three points be $x_{i-1}, \quad x_i, \quad x_{i+1}$. Derive differentiation formulas for the first and second derivatives at x_i. Simplify these expressions for uniformly spaced data with $\Delta = x_{i+1} - x_i$. You have just derived finite difference formulas for discrete data.

Problem 5 - Finite Difference Consider the central finite difference operator $\delta/\delta x$ defined by

$$\frac{\delta u_n}{\delta x} = \frac{u_{n+1} - u_{n-1}}{2h}$$

(a) In calculus we have

$$\frac{duv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Does the following analogous finite difference expression hold?

$$\frac{\delta(u_n v_n)}{\delta x} = u_n \frac{\delta v_n}{dx} + v_n \frac{\delta u_n}{\delta x}$$

(b) Show that

$$\frac{\delta(u_n v_n)}{\delta x} = \overline{u}_n \frac{\delta v_n}{\delta x} + \overline{v}_n \frac{\delta u_n}{\delta x}$$

where an overbar indicates average over the nearest neighbors,

$$\overline{u}_n = \frac{1}{2}(u_{n+1} + u_{n-1})$$

(c) Show that

$$\phi\frac{\delta\psi}{\delta x} = \frac{\delta}{\delta x}\bar{\phi}\psi - \overline{\psi\frac{\delta\phi}{\delta x}}$$

(d) Derive a finite difference formula for the second-derivative operator that is obtained from two applications of the first-derivative finite difference operator. Compare the leading error term of this formula and the popular second-derivative formula

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}$$

Use both schemes to calculate the second derivative of $sin(5x)$ at $x = 1.5$. Plot the absolute values of the errors as a function of h on a log-log plot. Use $10^{-4} \leq h \leq 10^0$. Discuss your plot.