Computation Exercises 7: Lambda Calculus Part 1

1. (**Free and bound variables.**)

   (a) i. Circle all the binding occurrences of variables in this λ-term:

   $$(\lambda\ x\ y\ .\ y\ (\lambda\ x\ .\ x\ y)\ z)(x\ (\lambda\ z\ x\ .\ x\ z\ y))$$

   ii. Circle all the bound occurrences of variables in this λ-term:

   $$(\lambda\ x\ y\ .\ y\ (\lambda\ x\ .\ x\ y)\ z)(x\ (\lambda\ z\ x\ .\ x\ z\ y))$$

   iii. Circle all the free occurrences of variables in this λ-term:

   $$(\lambda\ x\ y\ .\ y\ (\lambda\ x\ .\ x\ y)\ z)(x\ (\lambda\ z\ x\ .\ x\ z\ y))$$

   (b) Give the set of free variables for:

   i. $(\lambda x.\ xy)(x\lambda y.\ yx)(\lambda yz.\ zy)$

   ii. $(\lambda z.\ z(\lambda y.\ yzx)y)(\lambda xz.\ (\lambda\ y.\ zxy)x)$

2. (**α-Equivalence.**)

   (a) Which of the following λ-terms is α-equivalent to $(\lambda xy.\, y(\lambda x.\, xy)z)$?

   i. $(\lambda xy.\, a(\lambda x.\, xa)a)$

   ii. $(\lambda zy.\, y(\lambda x.\, xy)z)$

   iii. $(\lambda xy.\, y(\lambda z.\, zy)z)$

   iv. $(\lambda xy.\, y(\lambda z.\, zy)a)$

   v. $(\lambda xa.\, a(\lambda a.\, aa)z)$

   vi. $(\lambda xa.\, a(\lambda x.\, xa)a)$

   vii. $(\lambda xa.\, a(\lambda z.\, za)z)$

   viii. $(\lambda za.\, a(\lambda z.\, za)z)$

   (b) Write down three λ-terms which are α-equivalent to $(\lambda y.(\lambda x.\, xy)zxy)$.

   (c) For each of the three λ-terms you gave in part b, write down the set of free variables.

3. (**Expression substitution.**)
   Give the result of each of the following λ-term substitutions:

   (a) $(xy)[z/x]$

   (b) $(xy)[\lambda x.\ xx/x]$

   (c) $(\lambda x.\ xy)[z/y]$

   (d) $(\lambda x.\ xy)[z/x]$

   (e) $(\lambda x.\ xy)[x/y]$

   (f) $(\lambda x.\ xx)[\lambda x.\ xx/x]$

   (g) $(\lambda x.\ xy)[\lambda x.\ xy/y]$

   (h) $(\lambda x.\ xy)[x(\lambda x.\ xy)/y]$

4. (**β-reduction.**)

   (a) Perform a single β-reduction step on each of the following λ-terms. Give the whole derivation tree.

i. $(\lambda x.\, x)y$

ii. $(\lambda x.\, \lambda y.\, xy)y$

iii. $(\lambda x.\, \lambda y.\, xy)z$

iv. $\lambda x.x((\lambda x.\, x)y)$

(b) Find distinct λ-terms M, N such that M and N are *not* α-equivalent and:

$$((\lambda x.\, x)(\lambda x.\, xx))((\lambda x.\, x)(\lambda x.\, xx)) \to M$$

$$((\lambda x.\, x)(\lambda x.\, xx))((\lambda x.\, x)(\lambda x.\, xx)) \to N$$

(c) What happens if you continue reducing M and N?

(d) Let $T \triangleq \lambda x.xxx$. Perform some β-reduction steps on TT. There is no need to give full derivation trees. What do you observe?

5. (β**-normal-forms.**)
For each of the following λ-terms state whether it has a β-normal form and, if it has, find it.

(a) $(\lambda x.\ x)y$

(b) $y(\lambda x.\ x)$

(c) $(\lambda x.\ x)(\lambda y.\ y)$

(d) $(\lambda x.\ xx)(\lambda x.\ xx)$

(e) $(\lambda x.\ xx)(\lambda x.\ x)$

(f) $(\lambda x.\ x)(\lambda x.\ xx)$