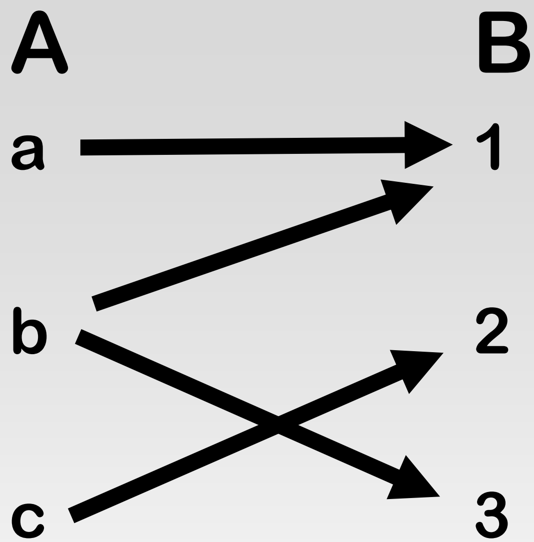
מתמטיקה בדידה
Discrete Math

Lecture 10

Last Week: Binary Relation

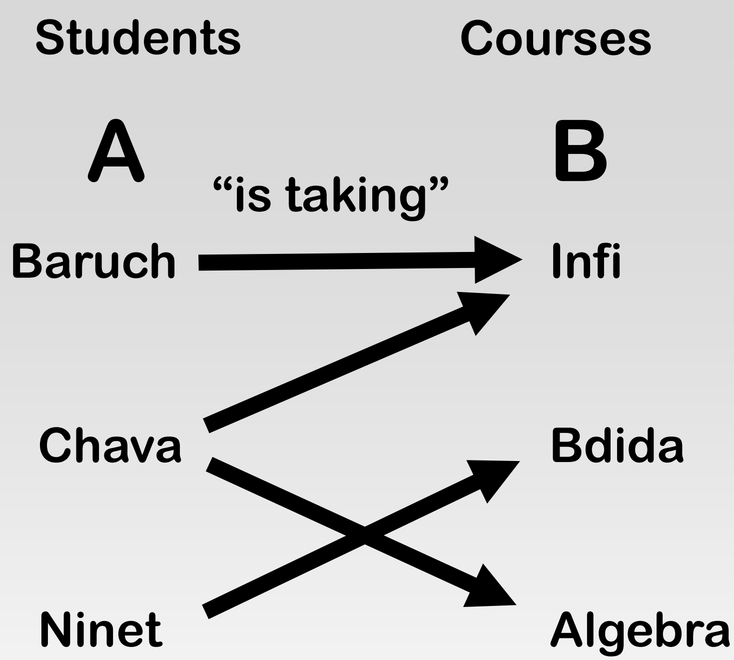


Describes relations between elements in A and elements in B.

Unlike a function: x $\in$ A can be related to more than one y $\in$ B.

Binary relation יחס בינארי

Binary Relation: Example



{(Baruch, infi), (Chava,Algebra), (Chava, Infi), (Ninet,Bdida)}

{ (1,a), (2,c), (2,a), (3,b) } (A = {1,2,3}, B = {a,b,c})

{ (1,1), (2,3), (2,1), (3,2) } (A = B ={1,2,3})

Binary Relations: Definition

Definition: A **binary relation**, R, consists of

1. a set, A, called the **domain** of R,
2. a set, B, called the **codomain** of R,
3. a subset $R \subseteq A \times B$ called the **graph** of R

Terminology:

1. We say that R is a relation between A and B.
2. If A = B, we say that R is a relation on A.

Notation:

1. aRb means that $(a,b) \in R$.
2. $a\not{R}b$ (alternatively, $\neg(aRb)$) means that $(a,b) \notin R$

Binary Relations: Characterization

Definition: A binary relation, R, on a set A is

Reflexive: $\forall a \in A$, aRa.

Anti-Reflexive: $\forall a \in A$, $\neg$(aRa).

Symmetric: $\forall a,b \in A$, aRb $\longrightarrow$ bRa.

Asymmetric: $\forall a,b \in A$, aRb $\longrightarrow$ $\neg$(bRa).

Anti-symmetric: $\forall a,b \in A$, (aRb $\wedge$ bRa) $\longrightarrow$ a = b.

Transitive: $\forall a,b,c \in A$, (aRb $\wedge$ bRc) $\longrightarrow$ aRc.

Asymmetric vs. Anti-Symmetric

Asymmetric aRb implies $\neg$(bRa) for all a,b $\in$ A.

Anti-symmetric aRb, bRa implies a = b for all a,b $\in$ A.

Anti-symmetric* aRb implies $\neg$(bRa) for all $a \neq b \in$ A.

Claim: Anti-symmetric $\equiv$ Anti-symmetric*

Can think of Anti-symmetric* as "weak Asymmetric"

Equivalence Relation

<u>Definition</u>: A binary relation, R, on a set A is said to be an <u>equivalence relation</u> if it is

Reflexive aRa for all a $\in$ A

Symmetric aRb implies bRa for all a,b $\in$ A

Transitive [aRb and bRc] implies aRc for all a,b,c $\in$ A

<u>Notation</u>: If a is equivalent to b, we write a ~ b.

Equivalence relation יחס שקילות

Equivalence Relations: Examples

"Equality" (=) – a ~ b if and only if a = b

"Same eye color" – a ~ b if and only if they have the same eye color.

"Same number of letters" – a ~ b are equivalent if and only if the number of letters in word a is the same as in b.

"Congruence mod 2" – a ~ b if and only if (a-b) is even.

Equivalence Class

Definition: Let R be an equivalence relation on A. The **equivalence class** of an element $a \in A$ is defined as:

$$[a]_R := \{b \in A \mid aRb\}$$

that is, the set of all elements in A that a is equivalent to.

Notation: Sometimes we write [a] instead of $[a]_R$

Equivalence class מחלקת שקילות

Equivalence Class: Examples

"Equality of sets" (=) – A ~ B if and only if A = B (as sets)

Q: What is the equivalence class [{1,2,3}]?

A: All sets whose elements are 1,2,3.

Examples: $\{1,3,2\} \in [\{1,2,3\}]$, $\{x \in \mathbb{N} \mid 0 < x \leq 3\} \in [\{1,2,3\}]$

"Same eye color" – a ~ b if and only if they have the same eye color.

Q: Yossi has blue eyes. What is [Yossi]?

A: All people with blue eyes.

"Congruence mod 2" – a ~ b if and only if (a-b) is even.

Q: What is [2]? What is [1]?

A: [2] = Evens, [1] = Odd numbers

Equivalence Class: Representatives

"Equality of sets" (=) – A ~ B if and only if A = B (as sets)

[{1,2,3}] = All sets whose elements are 1,2,3.

[{1,3,2}] = All sets whose elements are 1,2,3.

"Same eye color" – a ~ b if and only if they have the same eye color.

Yossi has blue eyes. [Yossi] = All people with blue eyes.

Ninet has blue eyes. [Ninet] = All people with blue eyes.

"Congruence mod 2" – a ~ b if and only if (a-b) is even.

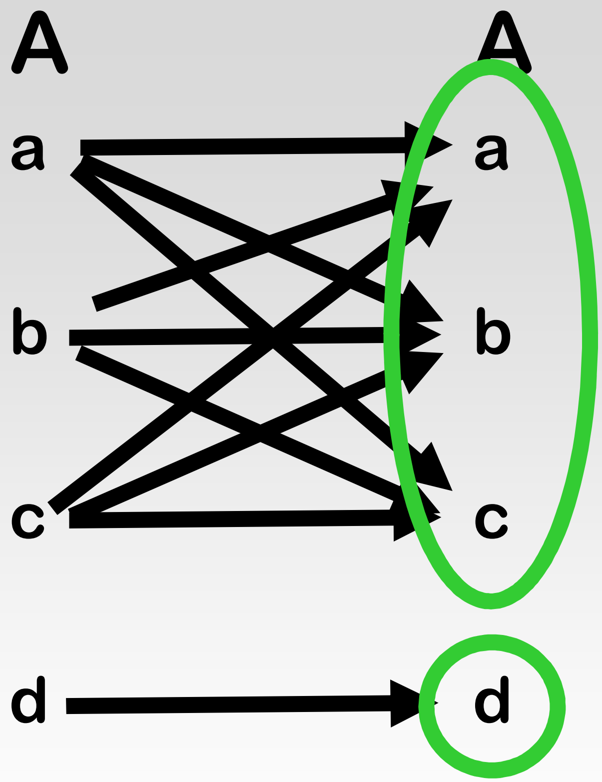
[2] = Evens, [1] = Odd numbers

[4] = Evens, [3] = Odd numbers

To describe an equivalence class $[a]_R$, it is sufficient to pick a representative in $[a]_R$

Representative נציג

Equivalence Relations



[a] = [b] = [c]

[x] := {y ∈ A | xRy}

[d]

Equivalence Class מחלקת שקילות

Representative נציג

For the Curious: The Rational Numbers

Elements in $\mathbb{Q}$ are though of as numbers a/b for a,b $\in \mathbb{Z}$.

But a/b = 2a/2b = 3a/3b, and so on…

So which one should we pick?

Also, how is a/b defined?

Define a relation R on $\mathbb{Z}^2$ in the following way:

$$R := \{((a,b),(c,d)) \; 2 \; \mathbb{Z}^2 x \mathbb{Z}^2 \mid ad=bc \}$$

That is, (a,b)R(c,d) if and only if ad = bc.

A <u>rational number</u> is simply a <u>representative</u> (a,b) of an equivalence class for the above relation R.

Exercise

We say that $a \in \mathbb{Z}$ is divisible by $b \in \mathbb{Z}$ if $\exists k \in \mathbb{Z}$ so that $a = kb$.
Define relations S,T on $\mathbb{Z}$ in the following way:

- iSj if and only if $i - j$ is divisible by 7.
- iTj if and only if $i + j$ is divisible by 7.

Q1: is S an equivalence relation?

Q2: is T an equivalence relation?

Q3: is $S \cup T$ an equivalence relation?

Partition

Definition: A **partition** of a set A is a collection of subsets $A_1, A_2, \ldots, A_n \subseteq A$ so that:

1. Pairwise disjoint: $\forall i,j \in [n],\ i \neq j \rightarrow A_i \cap A_j = \varnothing$
2. Covering: $A_1 \cup A_2 \cup \ldots \cup A_n = A$

Partition	חלוקה
Pairwise disjoint	זרות בזוגות
Covering	כיסוי

Partition: Examples

Let A be the set of all people.

1. Let A_1 be the set of all people with blue eyes
2. Let A_2 be the set of all people with green eyes
3. Let A_3 be the set of all people with brown eyes

   and so on…

Note that $A_1, A_2, \ldots, A_n \subseteq A$. Also:

1. Pairwise disjoint*: $\forall i,j \in [n],\ i \neq j \rightarrow A_i \cap A_j = \emptyset$
2. Covering**: $A_1 \cup A_2 \cup \ldots \cup A_n = A$

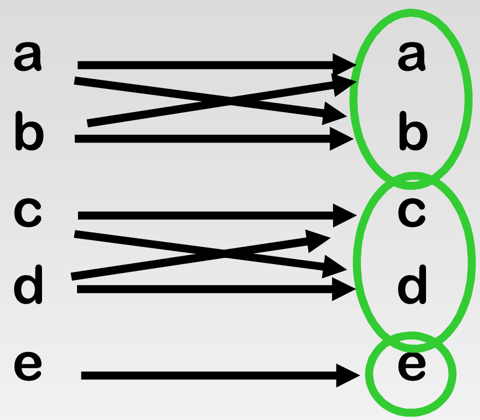
* Assuming there are no people with more than one eye color.

** Assuming we have used all eye colors.

Partition

Definition: A partition of a set A is a collection of subsets $A_1, A_2, \ldots, A_n \subseteq A$ so that:

1. Pairwise disjoint: $\forall i,j \in [n],\ i \neq j \rightarrow A_i \cap A_j = \emptyset$
2. Covering: $A_1 \cup A_2 \cup \ldots \cup A_n = A$



Partition and equivalence are the "same thing."

Partition	חלוקה
Pairwise disjoint	זרות בזוגות
Covering	כיסוי

Partition vs Equivalence

<u>Proposition</u>: Let A be a set. Then:

1. For any equivalence relation ~ on A, the collection:

$$\Omega = \{[a]_\sim \mid a \in A\}$$

forms a partition of A.

2. For any partition Ω of A, the relation:

$$R = \{(a,b) \in A \times A \mid \exists S \in \Omega, a \in S \text{ AND } b \in S\}$$

is an equivalence relation on A.

Induced Partition

Definition: Let ~ be an equivalence relation on A. then the partition:

$$\Omega = \{[a]_\sim \mid a \in A\}$$

is called the partition that is **induced** by ~.

Example: The partition on $\mathbb{Z}$ that is induced by the "Congruence mod 2" relation (a ~ b if and only if (a-b) is even) is:

A_1 = Evens,

A_2 = Odd numbers

Example 1

$A = \{1,2,3\} \times \{1,2,3\}$

$(x,y) \sim (x',y')$ if and only if $x+y = x'+y' \pmod 3$

$(1,1)\sim(2,3)\sim(3,2)$
$(1,2)\sim(2,1)\sim(3,3)$
$(2,2)\sim(1,3)\sim(3,1)$

$A_1 = \{(1,1),(2,3),(3,2)\}$
$A_2 = \{(1,2),(2,1),(3,3)\}$
$A_3 = \{(2,2),(1,3),(3,1)\}$

Example 2: Congruence mod 7

A = $\mathbb{N}$ and x~y if and only if x-y = 7k for some k $\in \mathbb{Z}$

1~8~15~22~…
2~9~16~23~…
3~10~17~24~…

A_1 = {n $\in \mathbb{N}$: n = 1+7k for some k $\in \mathbb{N}$} = [1]
A_2 = {n $\in \mathbb{N}$: n = 2+7k for some k $\in \mathbb{N}$} = [2]
A_3 = {n $\in \mathbb{N}$: n = 3+7k for some k $\in \mathbb{N}$} = [3]
and so on…

<u>Note</u>: $A_1 \cup A_2 \cup \ldots \cup A_7 = A$

$\forall i,j \in \{1,2,\ldots,7\},\ i \neq j \rightarrow A_i \cap A_j = \varnothing$

Partial Orders

Strict Partial Order

Definition: A binary relation, R, on a set A is said to be a strict partial order if it is

Asymmetric: $\forall a,b \in A$, $aRb \rightarrow \neg(bRa)$.

Transitive: $\forall a,b,c \in A$, $(aRb \wedge bRc) \rightarrow aRc$.

Teminology: A is said to be a partially ordered set (poset).

Notation: We use $\prec$ to denote a strict partial order R.

$a \prec b$ stands for aRb

The ordered pair $(A, \prec)$ denotes a poset.

Strict partial order	יחס סדר חלקי ממש
Partially ordered set	קבוצה סדורה חלקית (קס"ח)

Strict Partial Order: Examples

The < relation on numbers: a $\prec$ b iff a < b.

The $\subset$ relation on subsets: A $\prec$ B iff A $\subset$ B.

<u>Both examples are</u>:

<u>Asymmetric:</u> $\forall$a,b $\in$ A, aRb $\rightarrow$ $\neg$(bRa).

<u>Transitive:</u> $\forall$a,b,c $\in$ A, (aRb $\wedge$ bRc) $\rightarrow$ aRc.

Strict partial order — יחס סדר חלקי ממש

Partially ordered set — קבוצה סדורה חלקית (קס"ח)

Weak Partial Order

Definition: A binary relation, R, on a set A is said to be a weak partial order if it is

Reflexive: $\forall a \in A$, aRa.

Anti-symmetric*: $\forall a,b \in A$, $(aRb \wedge a \neq b) \longrightarrow \neg(bRa)$.

Transitive: $\forall a,b,c \in A$, $(aRb \wedge bRc) \longrightarrow aRc$.

Notation: We use $\preceq$ to denote a weak partial order R.

$a \preceq b$ stands for aRb

Weak partial order יחס סדר חלקי

Weak Partial Order: Examples

The $\leq$ relation on numbers: a$\preceq$ b iff a $\leq$ b.

The $\subseteq$ relation on subsets: A $\preceq$ B iff A $\subseteq$ B.

The "divides" relation. m $\preceq$ n iff $\exists$k so that n = km.

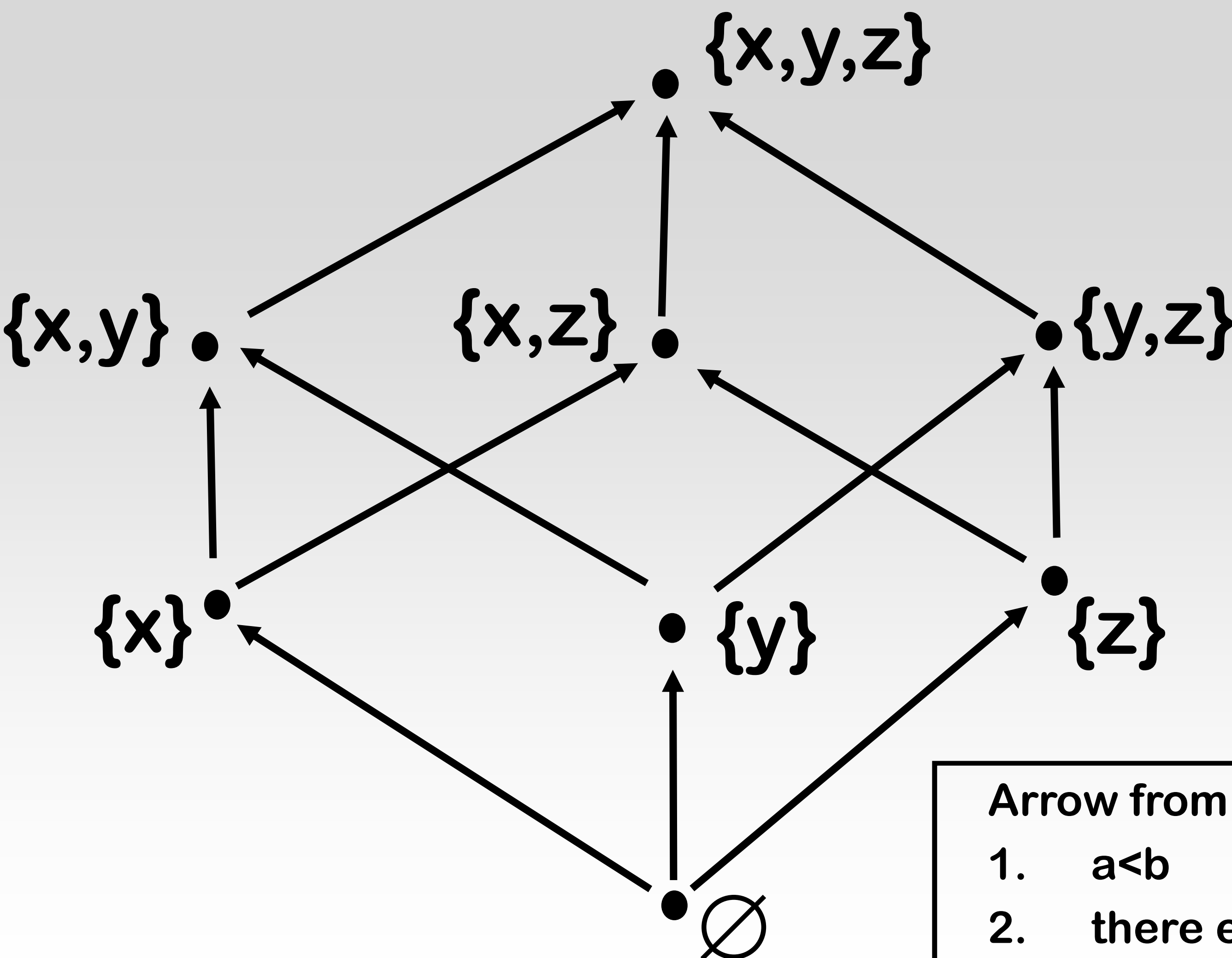
<u>All examples are</u>:

<u>Reflexive:</u> $\forall$a $\in$ A, aRa.

<u>Anti-symmetric*</u>: $\forall$a,b $\in$ A, (aRb $\wedge$ a $\neq$ b) $\longrightarrow$ $\neg$(bRa).

<u>Transitive:</u> $\forall$a,b,c $\in$ A, (aRb $\wedge$ bRc) $\longrightarrow$ aRc.

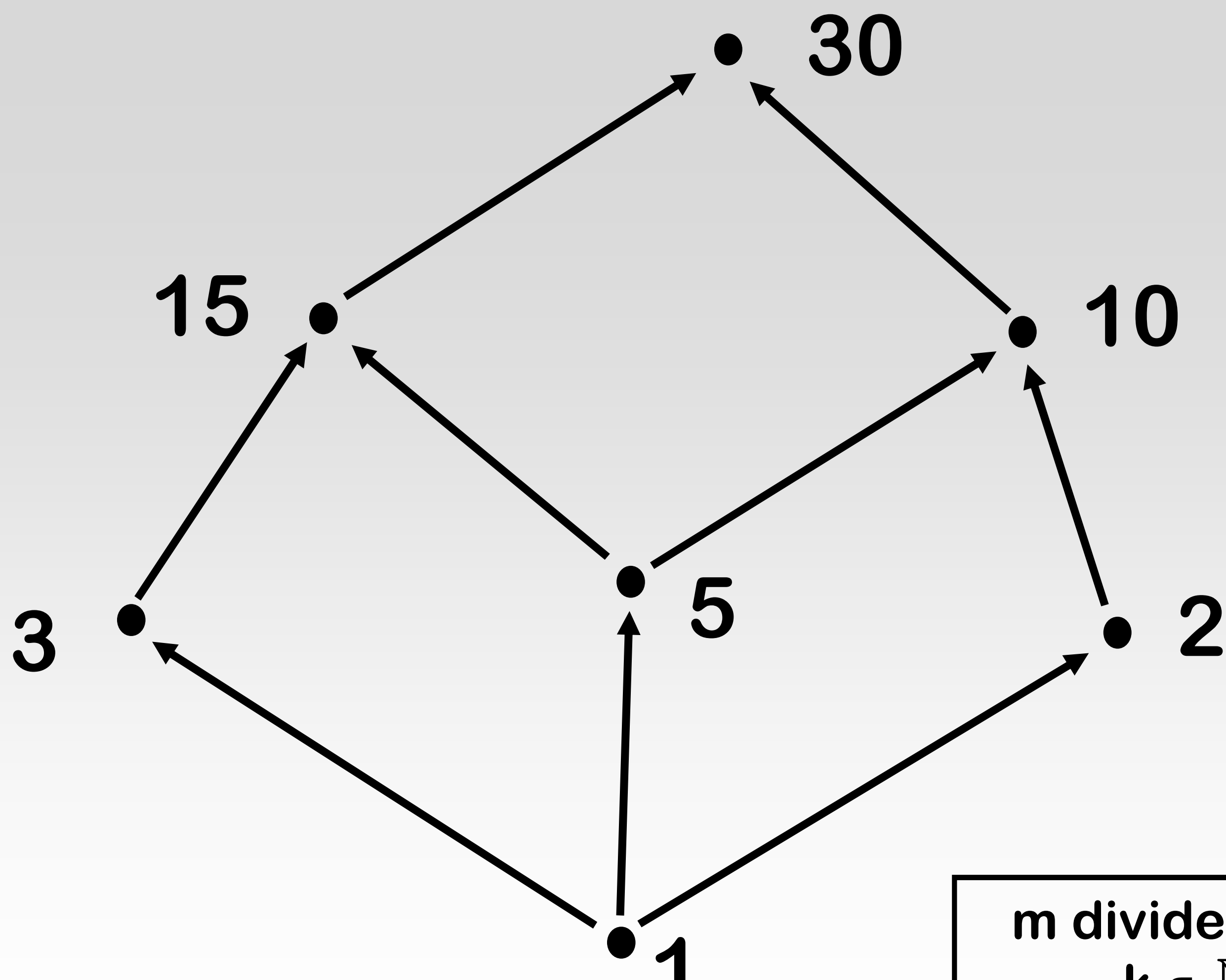
Hasse Diagram



Arrow from a to b if:

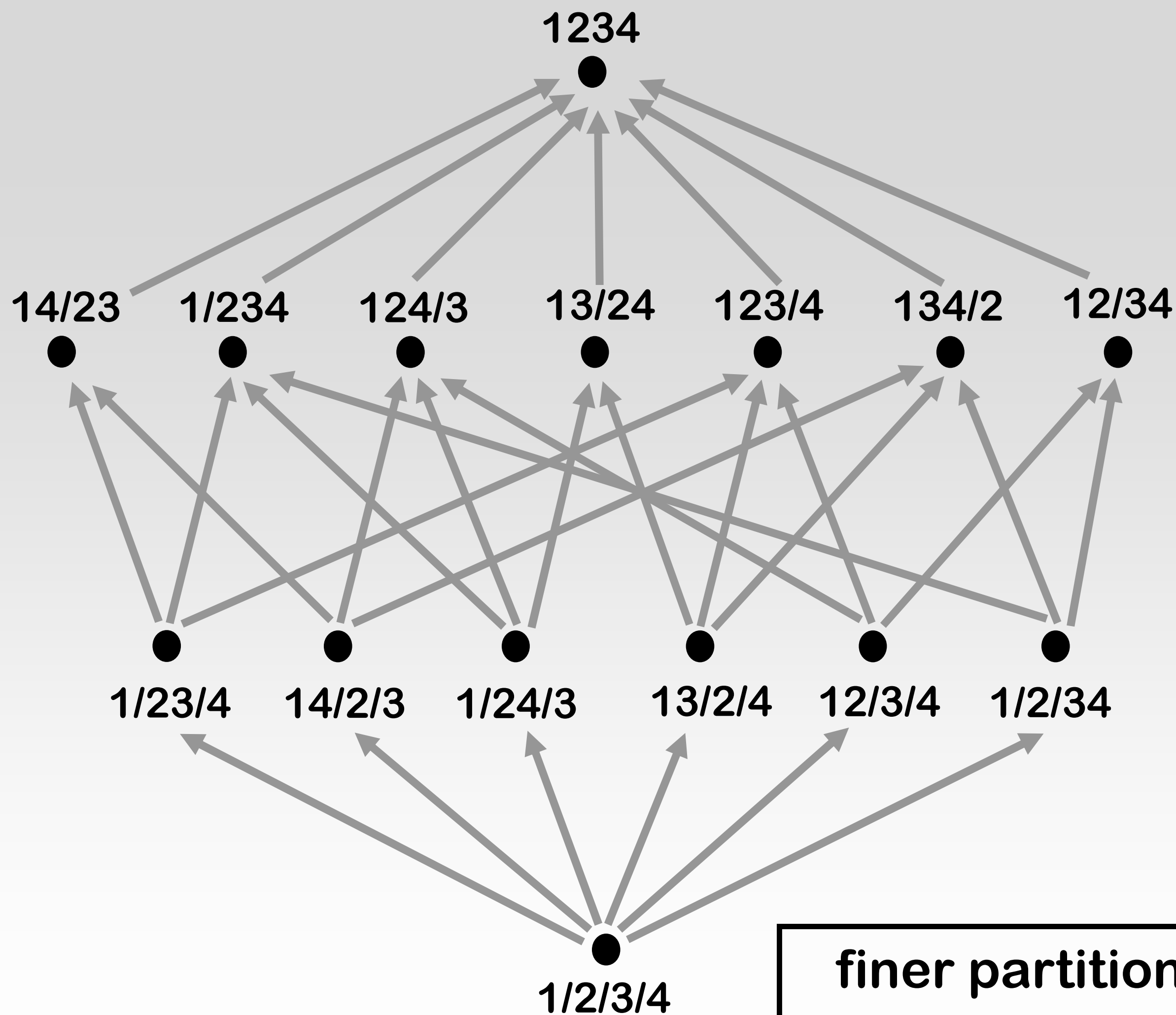
1. $a<b$
2. there exists no c such that $a<c<b$

Example: Divides Relation



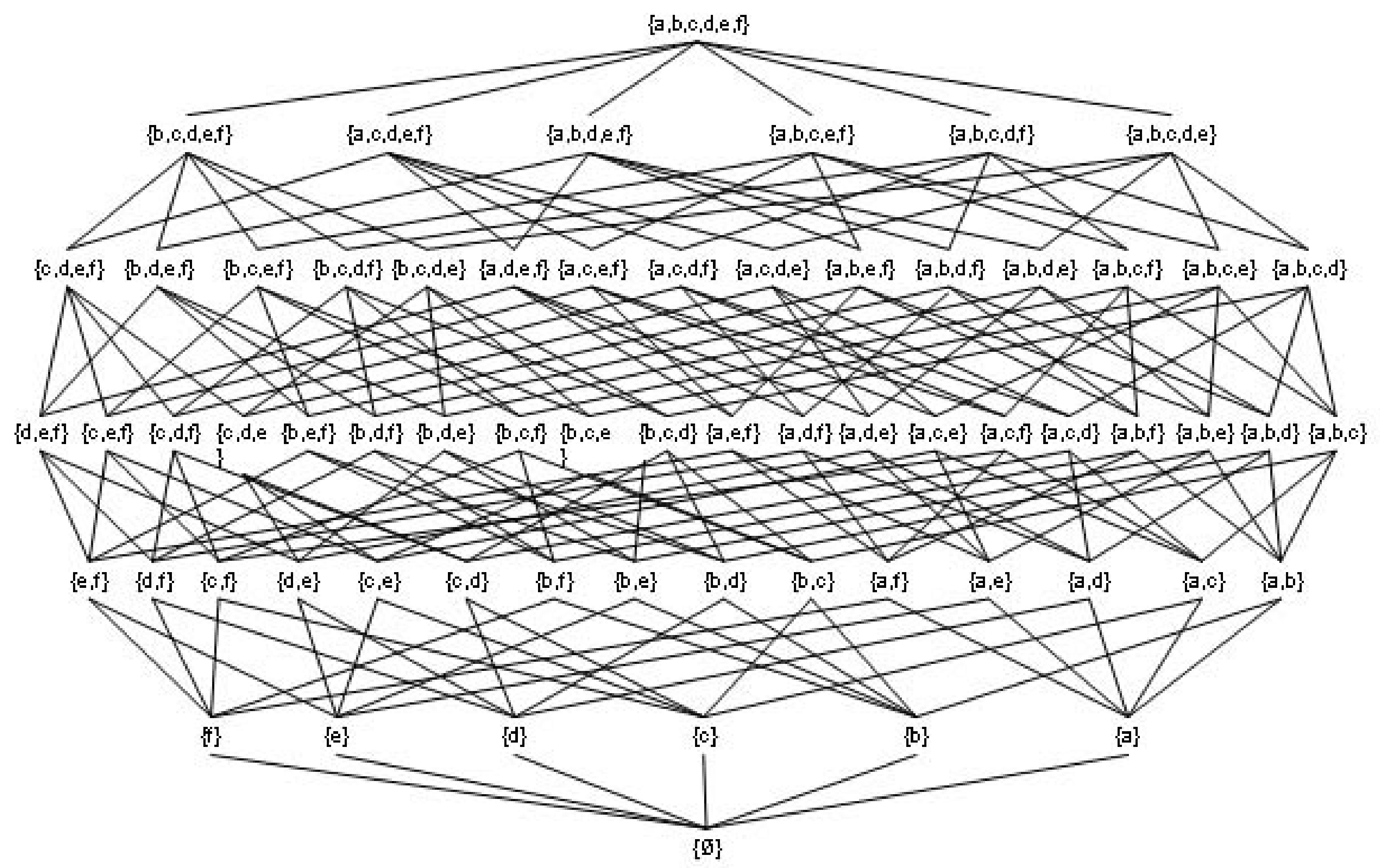
m divides n if there exists $k \in \mathbb{N}$ so that n=km

Example: Partitions of {1,2,3,4}



finer partition is < than coarser partition

Example: all Subsets of {a,b,c,d,e,f}



Taken from Wikipedia (where is the bug?)

Total Order

Definition: A partial order R is said to be total if

$$\forall a,b \in A,\ a \neq b \rightarrow (aRb) \text{ or } (bRa)$$

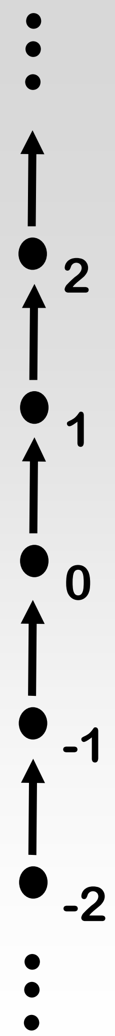
Every two different elements a,b $\in$ A are comparable.

Examples: The $\leq$, < relations on numbers.

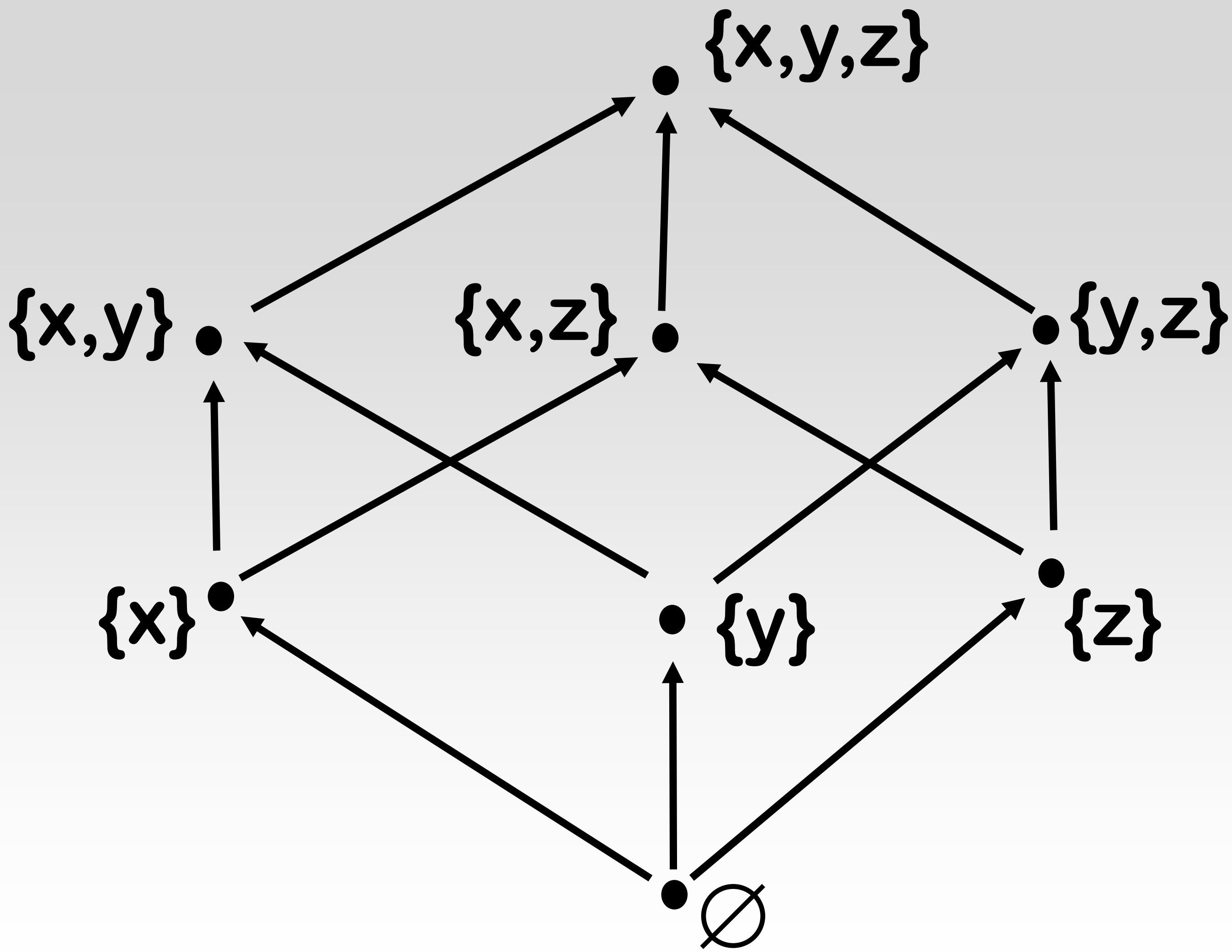
Non-examples: The $\subset$, $\subseteq$ relations on sets.

Comparable	ניתנים להשוואה/ברי השוואה
Total order	יחס סדר מלא

Example: < Relation on $\mathbb{Z}$



Non-Example: Subset Relation



Minimum, Minimal

<u>Definition</u>: Let be a partial order on a set A. An element a $\in$ A is <u>minimum</u> iff aRb for every other element b $\in$ A

<u>Definition</u>: Let be a partial order on a set A. An element a $\in$ A is <u>minimal</u> iff $\neg$(bRa) for every other element b $\in$ A.

<u>Note</u>:

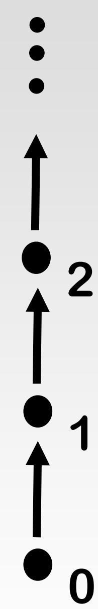
1. In a total order minimum and minimal are the same thing.
2. A partial order, however, may not have a minimum element and many minimal elements.
3. If a poset satisfies that every nonempty B $\subseteq$ A has a minimum, then is called a <u>totally ordered set</u>.

Totally ordered set קבוצה סדורה היטב

Example: < Relation on $\mathbb{N}$

Q: Is there a minimum?

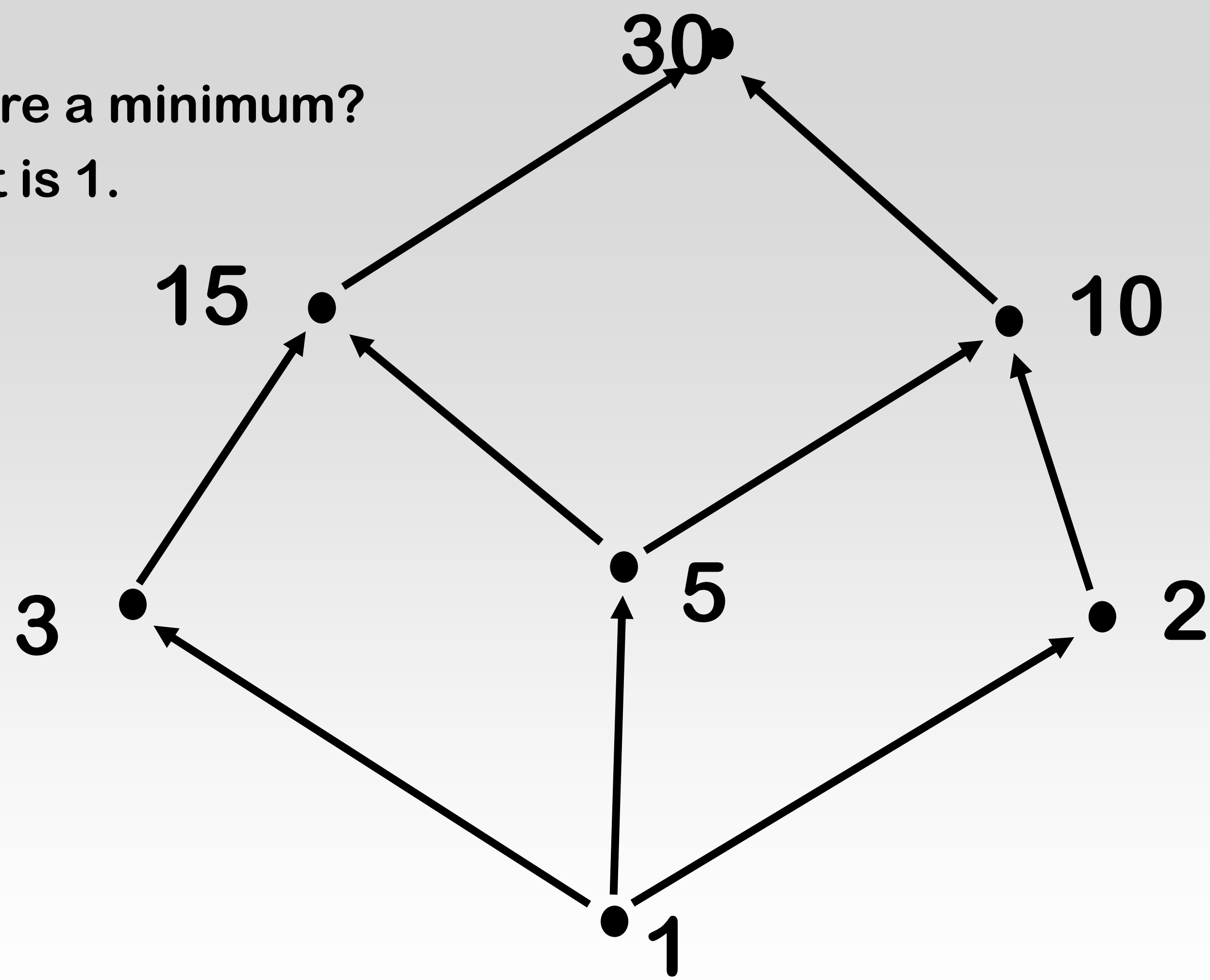
A: Yes. It is 0.



Example: Divides Relation

<u>Q</u>: Is there a minimum?

<u>A</u>: Yes. It is 1.



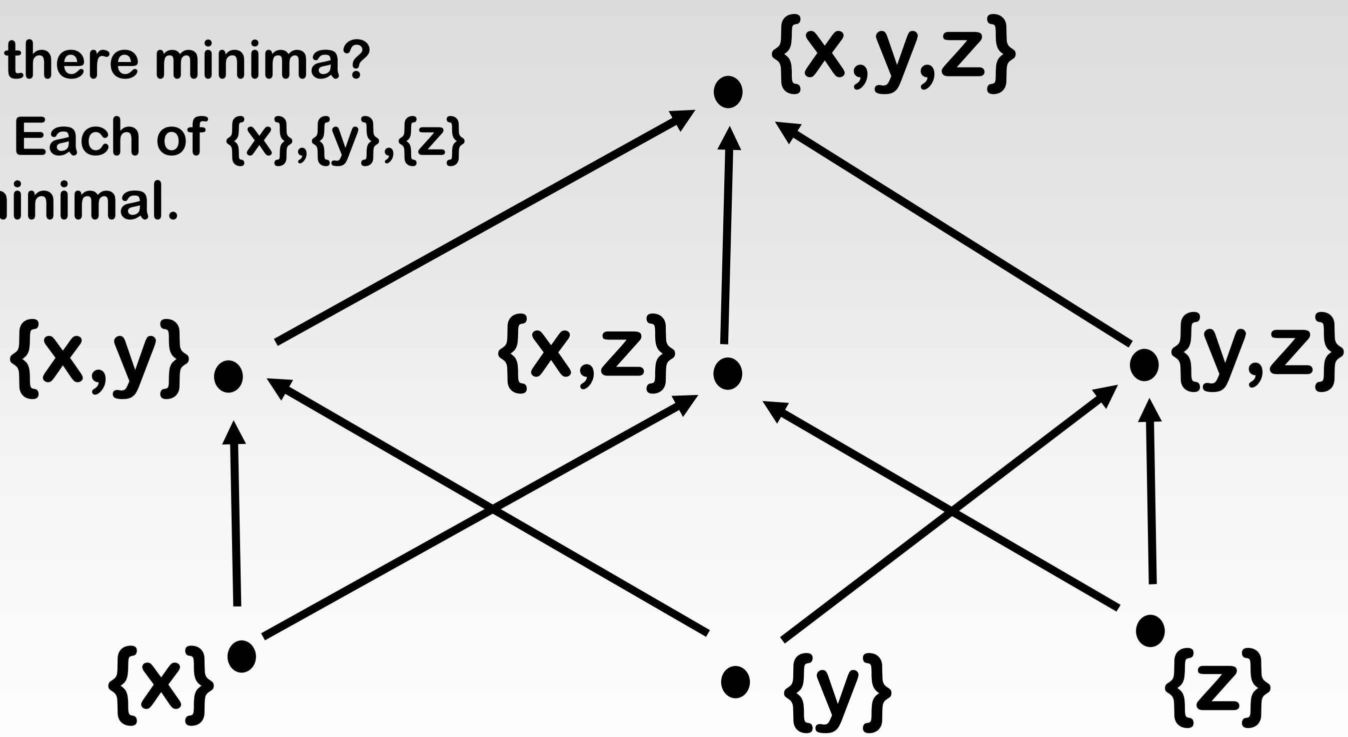
Example: Subset Relation on $P(\{x,y,z\})\setminus\emptyset$

Q1: Is there a minimum?

A1: No.

Q2: Are there minima?

A2: Yes. Each of {x},{y},{z} is minimal.



Exercise

Let A = {2,3,4,6,12}.

Let S = {(x, y) $\in$ A$\times$ A: x divides y}.

Q1: Find a total order relation on A that contains S.

Q2: Find a partial order relation on A, which is contained in S and has exactly three minimal elements.