# Rotations in 3D – The moment of inertia tensor

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### The angular momentum of a rigid body

$$\vec{L} = \sum_i \vec{r}_i \times m_i \vec{v}_i$$

and for every particle in a rigid body

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

so

$$\vec{L} = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

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### Digression: A vector triple product

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

This is so easy it can be done while riding in the **BAC**k of a **CAB**.

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### Back to the angular momentum

$$\vec{L} = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

can be rewritten as

$$\vec{L} = \sum_i m_i r_i^2 \vec{\omega} - \sum_i m_i \vec{r}_i (\vec{r}_i \cdot \vec{\omega})$$

The last term deserves a closer look.

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### Using matrix notation

Let's take a closer look at the term

$$\sum_i \vec{r}_i (\vec{r}_i \cdot \vec{\omega})$$

Dropping the $i$ for simplicity, and omitting the $m_i$ for the moment, if we write $\vec{\omega}$ as a column array, this term can also be written as follows:

$$\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} x(x\omega_x + y\omega_y + z\omega_z) \\ y(x\omega_x + y\omega_y + z\omega_z) \\ z(x\omega_x + y\omega_y + z\omega_z) \end{bmatrix}$$

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## Tensors

The matrix

$$\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}$$

is an instance of a **tensor of rank 2**, a generalization of the vector concept.

Because the elements of the matrix are formed by taking all the possible products of $x$, $y$, and $z$, it is written symbolically as

$$\vec{r}\vec{r}$$

This kind of tensor, that can be written as the matrix formed by taking all possible products between components of two vectors, is called a **dyadic tensor**, or simply **dyad**.



## Products of tensors and vectors

More generally, we can take two vectors $\vec{a}$ and $\vec{b}$, and construct the dyad

$$\mathbb{T} = \vec{a}\vec{b} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}$$

If we multiply to the right with the array $\vec{c}$

$$\begin{bmatrix} a_xb_x & a_xb_y & a_xb_z \\ a_yb_x & a_yb_y & a_yb_z \\ a_zb_x & a_zb_y & a_zb_z \end{bmatrix} \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} a_x(b_xc_x + b_yc_y + b_zc_z) \\ a_y(b_xc_x + b_yc_y + b_zc_z) \\ a_z(b_xc_x + b_yc_y + b_zc_z) \end{bmatrix}$$

This can be written symbolically as

$$(\vec{a}\vec{b}) \cdot \vec{c}$$

which is really the same as $\vec{a}(\vec{b} \cdot \vec{c})$.



## Products of tensors and vectors cont'd

Similarly, we can "dot multiply" a tensor $\mathbb{T} = \vec{a}\vec{b}$ by a vector to the left:

$$\vec{c} \cdot \mathbb{T} = \vec{c} \cdot (\vec{a}\vec{b}) = (\vec{c} \cdot \vec{a})\vec{b}$$

And why not multiply by a vector to the left and the right simultaneously?

$$\vec{c} \cdot \mathbb{T} \cdot \vec{d} = \vec{c} \cdot (\vec{a}\vec{b}) \cdot \vec{d} = (\vec{c} \cdot \vec{a})(\vec{b} \cdot \vec{d})$$

The result is a scalar.

## The identity tensor

The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the **unit tensor** $\mathbb{1}$, which satisfies for any $\vec{a}$

$$\mathbb{1} \cdot \vec{a} = \vec{a}$$

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## Back to the angular momentum – again

$$\vec{L} = \sum_i m_i r_i^2 \vec{\omega} - \sum_i m_i \vec{r}_i (\vec{r}_i \cdot \vec{\omega})$$

Now we can write this as

$$\vec{L} = \sum_i m_i r_i^2 \vec{\omega} - \sum_i m_i (\vec{r}_i \vec{r}_i) \cdot \vec{\omega}$$

or even better

$$\vec{L} = \sum_i m_i [r_i^2 \mathbb{1} - \vec{r}_i \vec{r}_i] \cdot \vec{\omega}$$

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## The moment of inertia tensor

$$\vec{L} = \sum_i m_i [r_i^2 \mathbb{1} - \vec{r}_i \vec{r}_i] \cdot \vec{\omega}$$

We can write this as

$$\vec{L} = \mathbb{I} \cdot \vec{\omega}$$

where $\mathbb{I}$ is the **moment of inertia** tensor,

$$\mathbb{I} = \sum_i m_i [r_i^2 \mathbb{1} - \vec{r}_i \vec{r}_i]$$

Clearly, the angular momentum need not be parallel to the rotation axis generally.

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