



Week of Jan 28-Feb3

Multipole Expansion, Electric Fields in Matter

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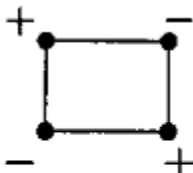
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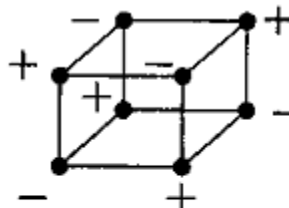


Multipoles


Monopole
($V \sim 1/r$)


Dipole
($V \sim 1/r^2$)

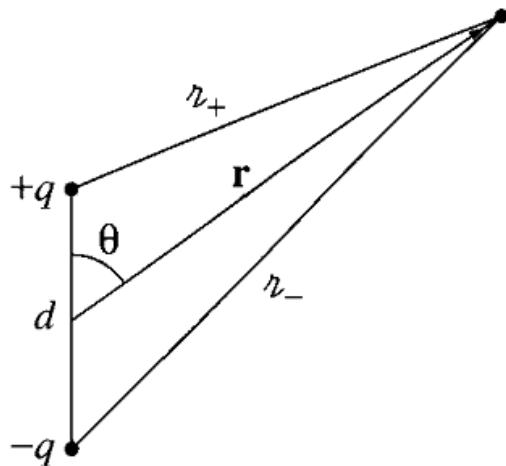

Quadrupole
($V \sim 1/r^3$)


Octopole
($V \sim 1/r^4$)

Disclaimer: Most of the slides and figures in this PPT are taken from –
'Introduction to electrodynamics'-D J Griffiths



Dipole



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_+} - \frac{q}{r_-} \right)$$

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0} \frac{qd \cos \theta}{r^2}$$

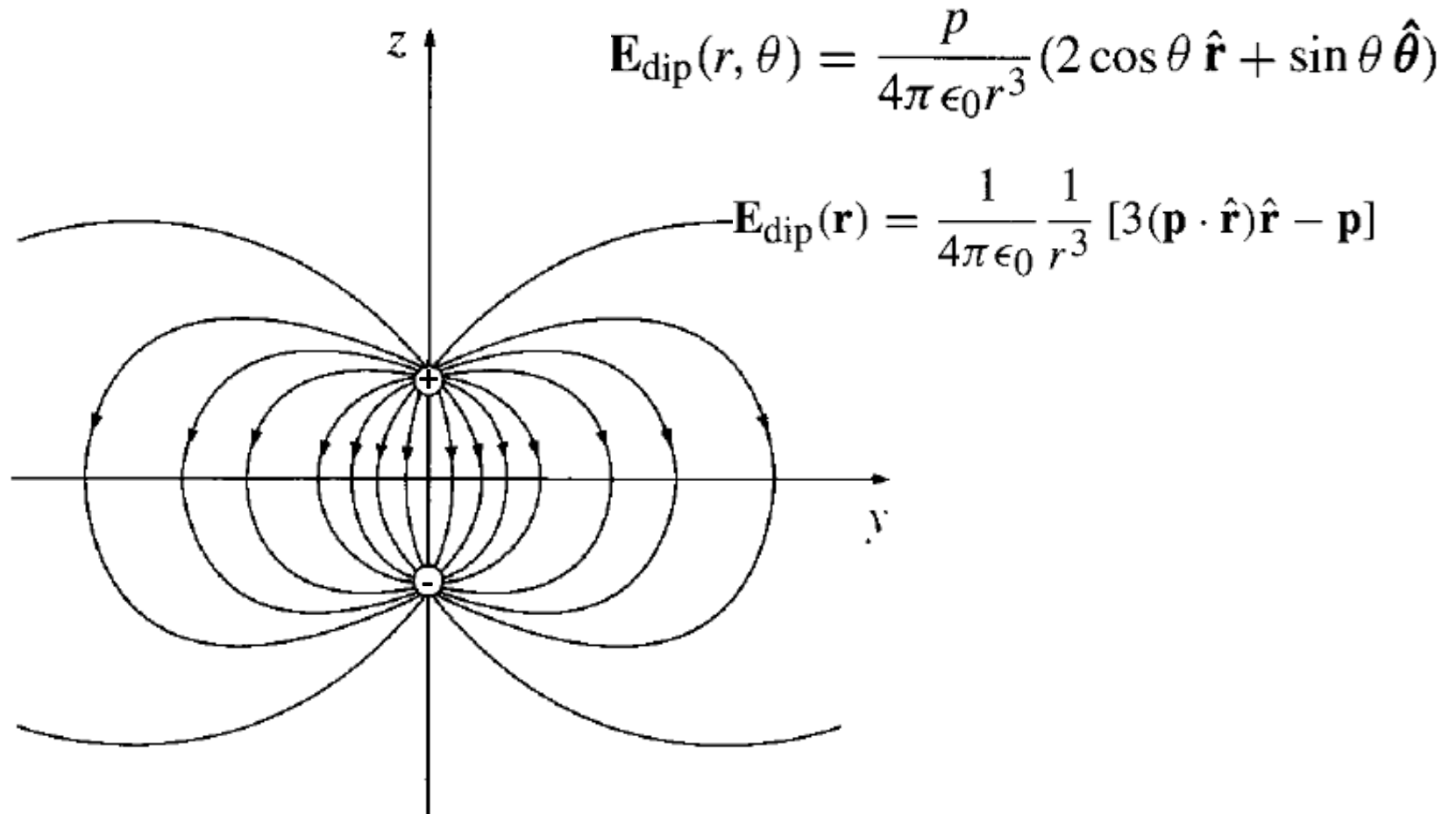
$$\mathbf{E}_{\text{dip}}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}})$$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]$$

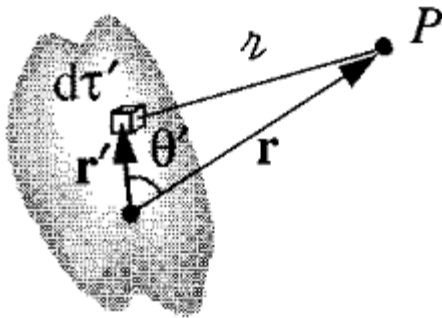


Field of a dipole





Multipole expansion



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{z} \rho(\mathbf{r}') d\tau'$$

$$\frac{1}{z} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos \theta')$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos \theta') \rho(\mathbf{r}') d\tau'$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{1}{r} \int \rho(\mathbf{r}') d\tau' + \frac{1}{r^2} \int r' \cos \theta' \rho(\mathbf{r}') d\tau' + \frac{1}{r^3} \int (r')^2 \left(\frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) \rho(\mathbf{r}') d\tau' + \dots \right]$$



Dipole component- generic expression

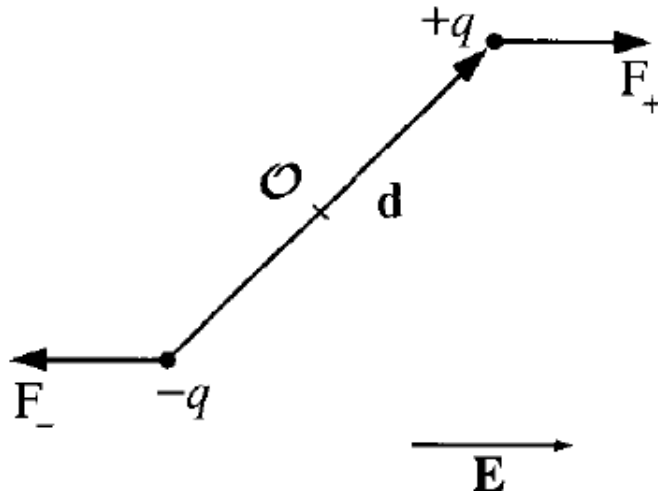
$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \hat{\mathbf{r}} \cdot \int \mathbf{r}' \rho(\mathbf{r}') d\tau'$$

$$\mathbf{p} \equiv \int \mathbf{r}' \rho(\mathbf{r}') d\tau'$$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}.$$



Torque and Force on a dipole



$$\mathbf{N} = (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-)$$

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}.$$

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q(\mathbf{E}_+ - \mathbf{E}_-) = q(\Delta \mathbf{E})$$

$$\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E}.$$



Energy of a dipole

$$dU = \tau d\theta$$

$$\tau = pE \sin \theta$$

$$\begin{aligned} U(\theta) - U(0) &= \int_0^\theta pE \sin \theta d\theta \\ &= -pE \cos \theta + pE \end{aligned}$$

$$U = -\mathbf{p} \cdot \mathbf{E}$$



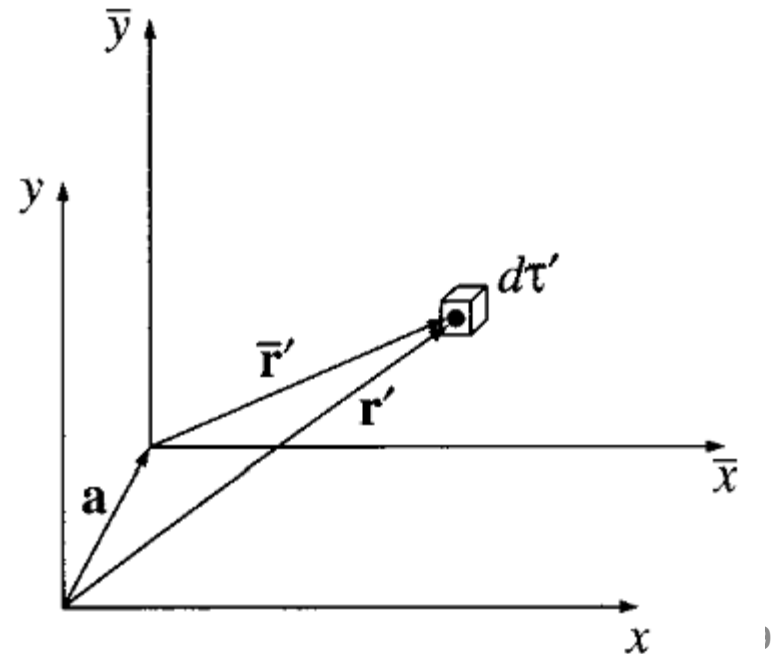
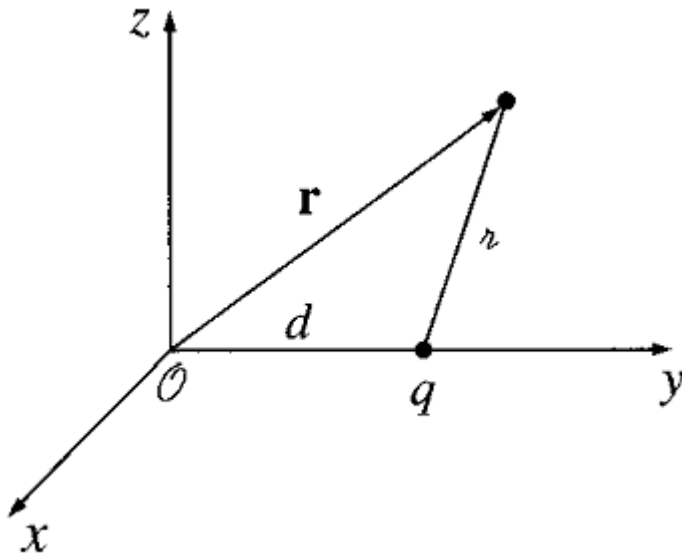
Generic dipole moment

$$\mathbf{p} = \sum_{i=1}^n q_i \mathbf{r}'_i$$

$$\bar{\mathbf{p}} = \int \bar{\mathbf{r}}' \rho(\mathbf{r}') d\tau' = \int (\mathbf{r}' - \mathbf{a}) \rho(\mathbf{r}') d\tau'$$

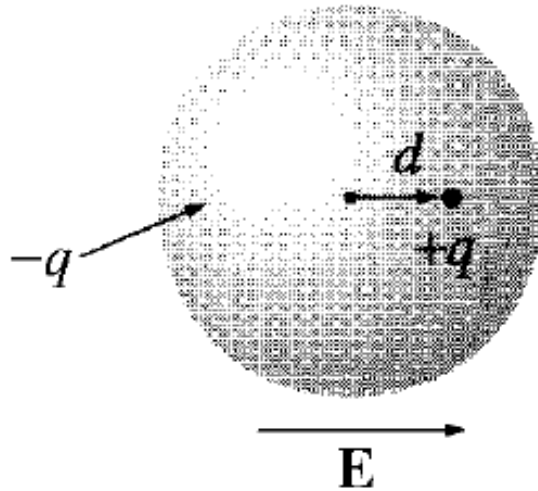
$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

$$= \mathbf{p} - Q\mathbf{a}$$





Electric field in a media



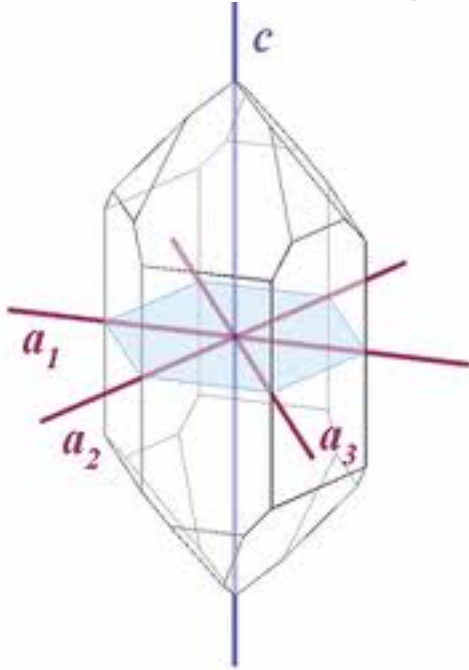
$$\mathbf{p} = \alpha \mathbf{E}$$

Anisotropic media

$$p_x = \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z$$

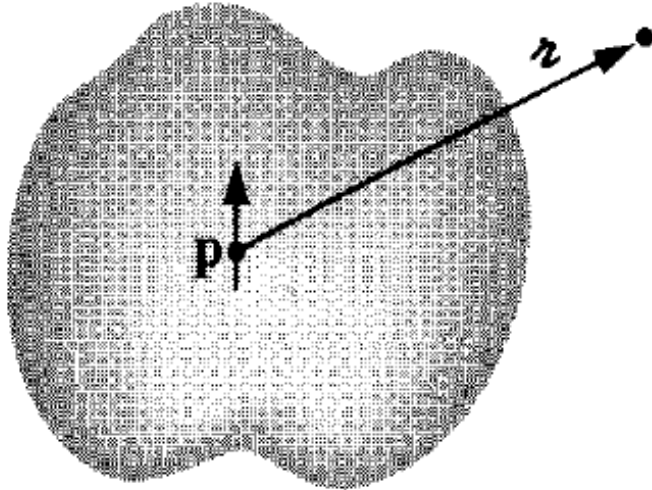
$$p_y = \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z$$

$$p_z = \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z$$





Field of a polarized object



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}} \cdot \mathbf{p}}{r^2}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\hat{\mathbf{r}} \cdot \mathbf{P}(\mathbf{r}')}{r^2} d\tau'.$$

$$V = \frac{1}{4\pi\epsilon_0} \oint_S \frac{1}{r} \mathbf{P} \cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0} \int_V \frac{1}{r} (\nabla' \cdot \mathbf{P}) d\tau'$$



Bound charges

$$\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}} \qquad \rho_b \equiv -\nabla \cdot \mathbf{P}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_S \frac{\sigma_b}{r} da' + \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_b}{r} d\tau'$$

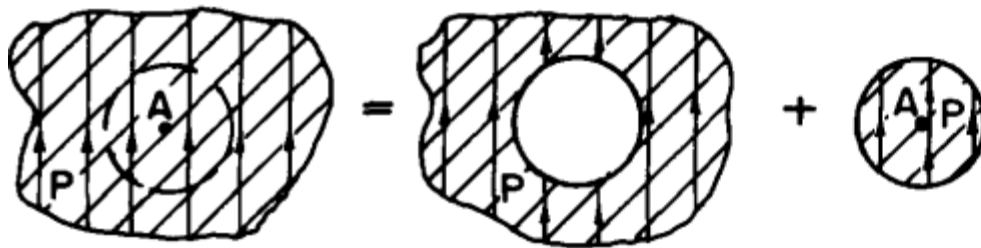


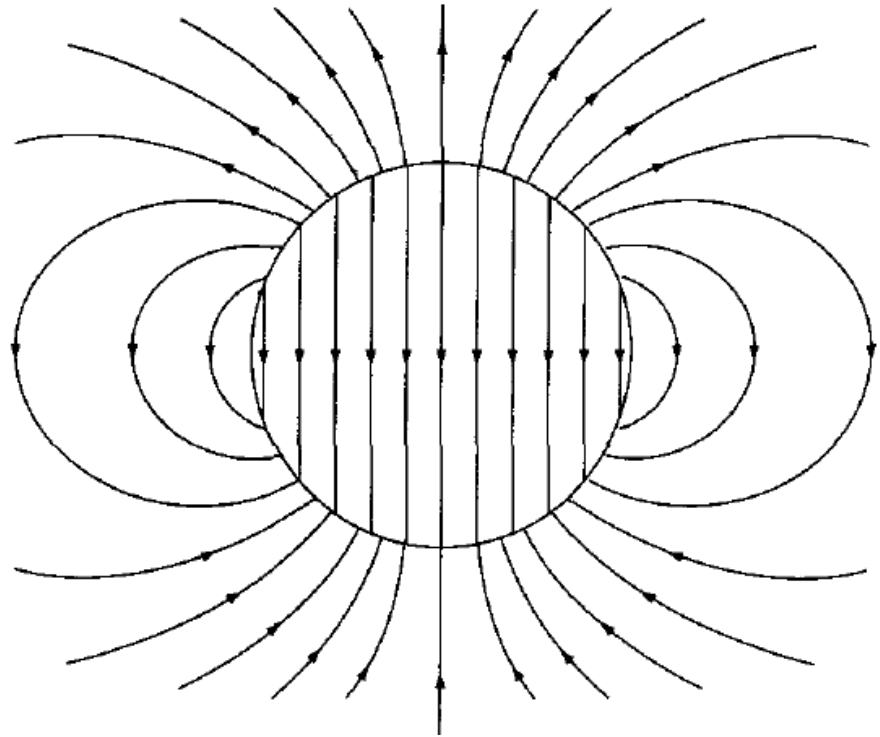
Fig. 11-6. The field at any point A in a dielectric can be considered as the sum of the field in a spherical hole plus the field due to a spherical plug.



Uniformly polarized dielectric

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P \cos \theta$$

$$V(r, \theta) = \begin{cases} \frac{P}{3\epsilon_0} r \cos \theta, & \text{for } r \leq R \\ \frac{P}{3\epsilon_0} \frac{R^3}{r^2} \cos \theta, & \text{for } r \geq R \end{cases}$$



$$r < R$$

$$\mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0} \hat{\mathbf{z}} = -\frac{1}{3\epsilon_0} \mathbf{P}$$

$$r > R$$

$$\mathbf{E}_{\text{dip}}(r, \theta) = \frac{P}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}})$$

Force on a dielectric

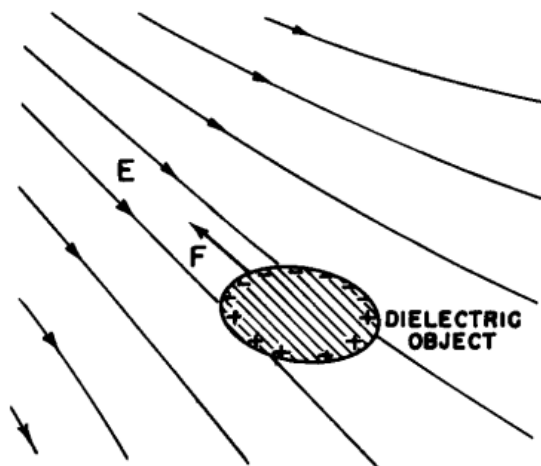
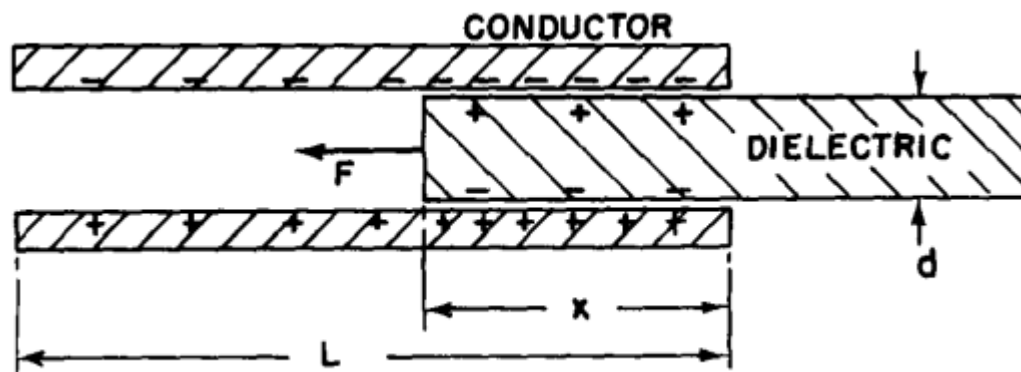


Fig. 10-8. A dielectric object in a nonuniform field feels a force toward regions of higher field strength.



$$F_x = -\frac{\partial U}{\partial x} = +\frac{V^2}{2} \frac{\partial C}{\partial x}$$



Electric Displacement- Gauss Law

$$\rho = \rho_b + \rho_f$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$$

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f\text{enc}}$$



Linear dielectrics

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E}$$

$$\mathbf{D} = \epsilon \mathbf{E}$$

$$\epsilon \equiv \epsilon_0 (1 + \chi_e)$$

permittivity of the medium

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

relative permittivity of the medium

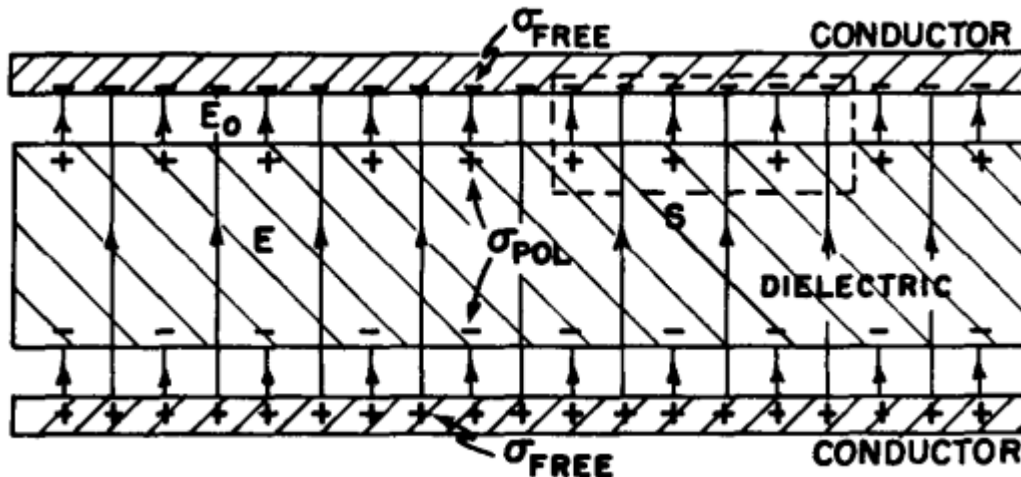
Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7
Neon	1.00013	Salt	5.9
Hydrogen	1.00025	Silicon	11.8
Argon	1.00052	Methanol	33.0
Air (dry)	1.00054	Water	80.1
Nitrogen	1.00055	Ice (-30° C)	99
Water vapor (100° C)	1.00587	KTaNbO ₃ (0° C)	34,000



Boundary conditions- Dielectric

$$D_{\text{above}}^{\perp} - D_{\text{below}}^{\perp} = \sigma_f$$

$$D_{\text{above}}^{\parallel} - D_{\text{below}}^{\parallel} = P_{\text{above}}^{\parallel} - P_{\text{below}}^{\parallel}$$





Boundary Value Problem with linear Dielectrics

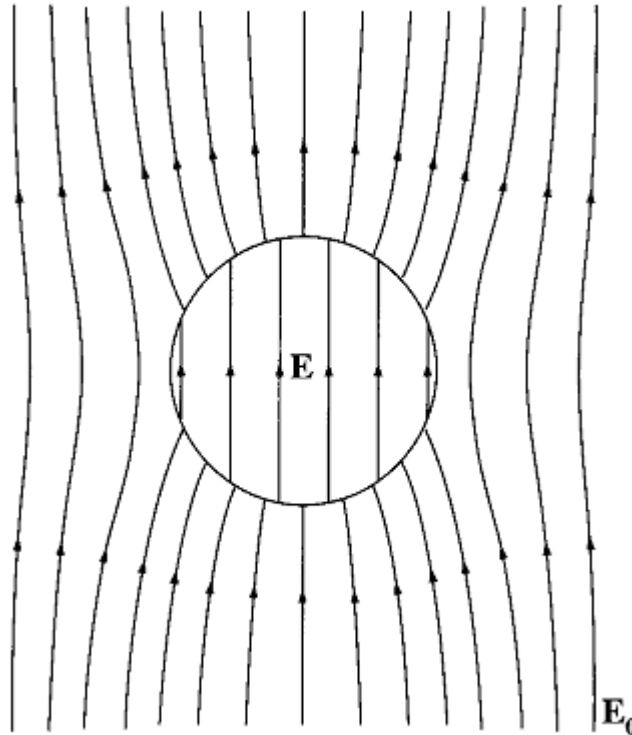
$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\epsilon_0 \frac{\chi_e}{\epsilon} \mathbf{D} \right) = - \left(\frac{\chi_e}{1 + \chi_e} \right) \rho_f$$

$$\epsilon_{\text{above}} E_{\text{above}}^{\perp} - \epsilon_{\text{below}} E_{\text{below}}^{\perp} = \sigma_f$$

$$\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f$$



Linear dielectric in a Uniform field



Solve Laplace Equation with Boundary Conditions

$$(i) \quad V_{\text{in}} = V_{\text{out}}, \quad \text{at } r = R,$$

$$(ii) \quad \epsilon \frac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \frac{\partial V_{\text{out}}}{\partial r}, \quad \text{at } r = R,$$

$$(iii) \quad V_{\text{out}} \rightarrow -E_0 r \cos \theta, \quad \text{for } r \gg R$$

$$V_{\text{in}}(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

$$V_{\text{out}}(r, \theta) = -E_0 r \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

$$V_{\text{in}}(r, \theta) = -\frac{3E_0}{\epsilon_r + 2} r \cos \theta = -\frac{3E_0}{\epsilon_r + 2} z$$

$$\mathbf{E} = \frac{3}{\epsilon_r + 2} \mathbf{E}_0$$



Energy in Dielectrics

$$C = \epsilon_r C_{\text{vac}} \quad W = \frac{1}{2} C V^2$$

$$W = \frac{\epsilon_0}{2} \int \epsilon_r E^2 d\tau = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} d\tau$$

$$W_{\text{tot}} = W_{\text{free}} + W_{\text{bound}} + W_{\text{spring}}$$