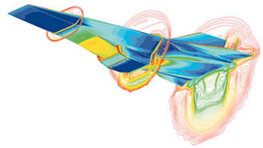
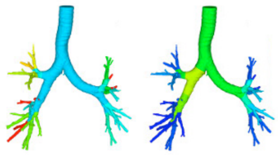


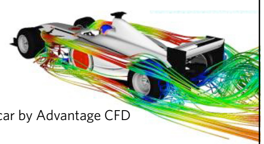
Hyper-X at Mach 7, NASA image in the public domain

ME 702
Computational Fluid Dynamics, CFD

Prof. Lorena A. Barba



Human airways, by FuiDA nv



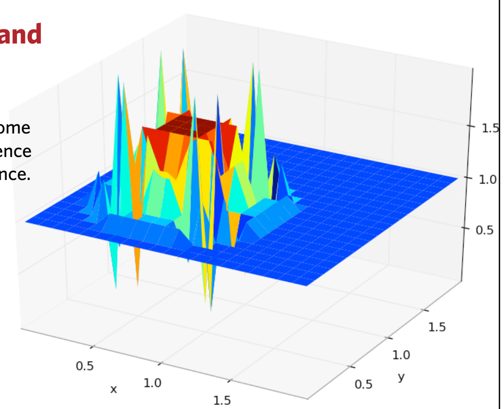
F1 car by Advantage CFD

BU

Consistency, Stability, and error analysis

... from TED-Ed of video lesson 8, some confusion remains about the difference between consistency and convergence.

http://ed.ted.com/on/jO71snfk



- DEF— **Consistency**

  A condition on the numerical scheme, namely that the scheme must tend to the differential equation when the steps in time and space tend to zero.

- DEF— **Convergence**

  A condition on the numerical solution, that it should tend to the exact solution of the mathematical model, when time and space steps tend to zero (i.e., when the mesh is refined).

- A consistent scheme is one in which the truncation error goes to zero ... $$\lim_{\Delta t, \Delta x \to 0} \epsilon_T = 0$$

- e.g., for $u_t + c\, u_x = 0$
  - the scheme forward-time/centered space is consistent, but unstable
  - solutions do not converge

- Convergence: $\tilde{\epsilon}_T = u_i^n - \tilde{u}_i^n$ $$\lim_{\Delta t, \Delta x \to 0} \tilde{\epsilon}_T = 0$$

- Another example: $u_t = \nu u_{xx}$

  discretized with central difference in time:

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{\nu}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

leapfrog

  - This scheme is called Richardson's method. It is unstable.

- make a stable scheme using $u_i^n = \frac{1}{2}\left(u_i^{n+1} + u_i^{n-1}\right)$
  what happens?

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{\nu}{\Delta x^2}\left(u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n\right)$$

- using Taylor expansions, we can show that the errors are:

$$O(\Delta x^2),\, O(\Delta t^2),\, O(\Delta t^2/\Delta x^2)$$

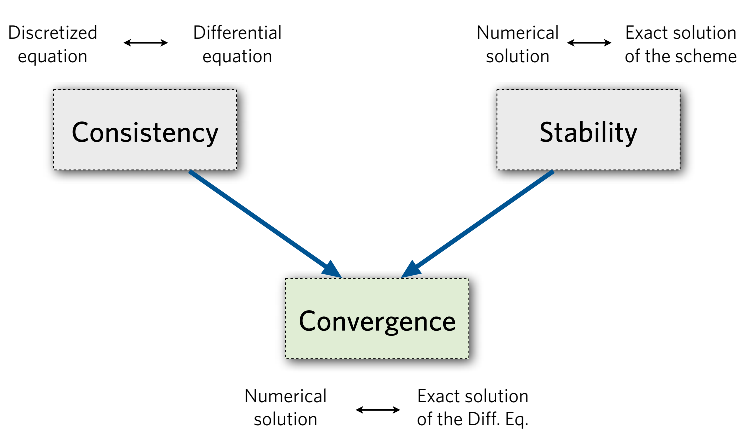
- It tends to: $$u_t + \nu \frac{\Delta t^2}{\Delta x^2} u_{tt} = \nu u$$

- It is inconsistent!

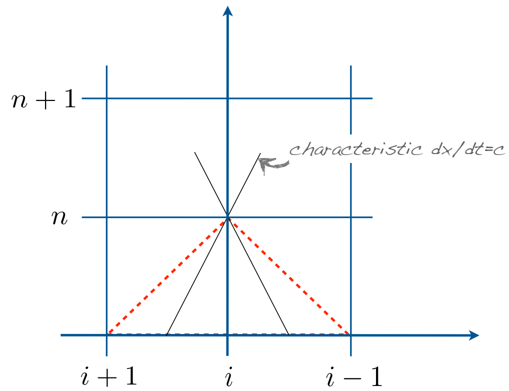
- TED-Ed student question:
  - One of the notes for stability said that the stability condition is a requirement on the numerical "scheme" only - it does not require any condition on the differential equation. Are numerical "scheme" and "solution" being used interchangeably here? Earlier we said that stability is a condition on the numerical solution, whereas consistency is a condition on the numerical scheme.

Equivalence theorem of Lax

For a well-posed I.V.P. and a consistent discretization scheme, stability is the necessary and sufficient condition for convergence.



- Physical interpretation of the CFL condition



von Neumann stability analysis

Recall "Lax equivalence theorem"

verify consistency
establish stability } convergence ensured

- Investigate propagation & amplification of errors:
  - Assume: linear PDE, periodic BCs $$u_i^n = \sum_{j=-N}^{N} V_j^n \, e^{Ik_j x_i}$$