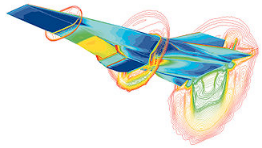
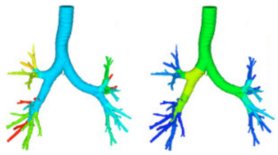


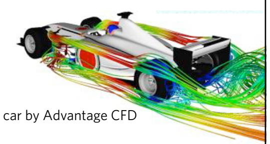
Hyper-X at Mach 7, NASA image in the public domain

ME 702
Computational Fluid Dynamics, CFD

Prof. Lorena A. Barba



Human airways, by FuiDA nv



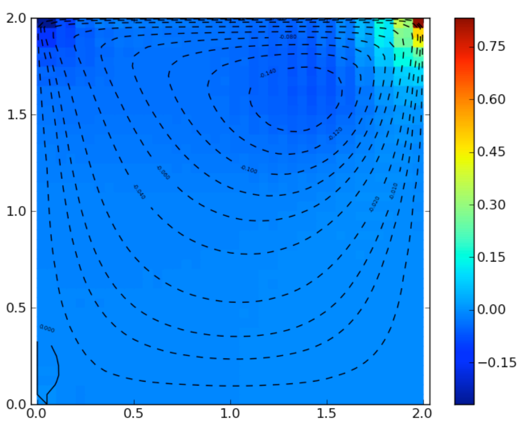
F1 car by Advantage CFD

BU

Navier-Stokes **Cavity Flow**

- Run your code with:

- a 41 x 41 mesh, dx = dy = 0.05
- viscosity = 0.01 (giving Re=200)
- high-frequency oscillations on the pressure!



Navier-Stokes **Discretized 1D Equations**

- continuity

$$\frac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{2\Delta x}=0$$

- momentum

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+\frac{1}{2}\frac{(u_{i+1}^n)^2-(u_{i-1}^n)^2}{2\Delta x}=-\frac{1}{\rho}\frac{p_{i+1}-p_{i-1}}{2\Delta 2}+\nu\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{\Delta x^2}$$

- pressure correction:
  - step 1

$$\frac{u_i^\star-u_i^n}{\Delta t}+\frac{1}{2}\frac{(u_{i+1}^n)^2-(u_{i-1}^n)^2}{2\Delta x}=\nu\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{\Delta x^2}$$

  - step 2

$$\frac{u_i^{n+1}-u_i^\star}{\Delta t}=-\frac{1}{\rho}\frac{p_{i+1}-p_{i-1}}{2\Delta x}$$

Navier-Stokes **Pressure Equation**

- step 2 gives:

$$u_i^{n+1} = -\frac{\Delta t}{\rho}\frac{p_{i+1} - p_{i-1}}{2\Delta x} + u_i^\star$$

- use this expression in the discretized continuity equation:

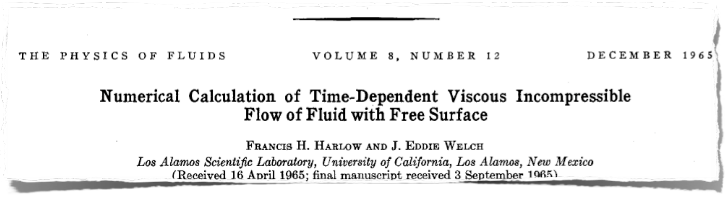
$$\frac{1}{2\Delta x}\left(\frac{-\Delta t}{\rho}\frac{p_{i+2} - p_i}{2\Delta x} + u_{i+1}^\star + \frac{\Delta t}{\rho}\frac{p_i - p_{i-2}}{2\Delta x} + u_{i-1}^\star\right) = 0$$

- rewrite

$$\boxed{\frac{p_{i+2} - 2p_i + p_{i-2}}{4\Delta x^2} = \frac{\rho}{\Delta t}\frac{u_{i+1}^\star - u_{i-1}^\star}{2\Delta x} = 0}$$

Navier-Stokes **Solution to the problem**

- "staggered grid" $\Longrightarrow$ due to Harlow & Welch (1965)


THE PHYSICS OF FLUIDS VOLUME 8, NUMBER 12 DECEMBER 1965

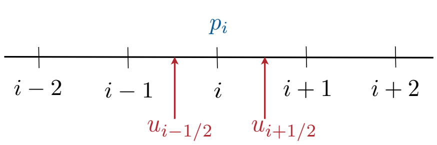
Numerical Calculation of Time-Dependent Viscous Incompressible Flow of Fluid with Free Surface

Francis H. Harlow and J. Eddie Welch

Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico

(Received 16 April 1965; final manuscript received 3 September 1965)

3335 citations on Google scholar, checked 8 Mar.'12



Navier-Stokes **Staggered-grid equations**

- continuity $\dfrac{u_{i+1/2}^{n+1} - u_{i-1/2}^{n+1}}{\Delta x} = 0$

- with a fractional step, we have $\dfrac{u_{i+1/2}^{n+1} - u_{i+1/2}^*}{\Delta t} = -\dfrac{1}{\rho}\dfrac{p_{i+1} - p_i}{\Delta x}$

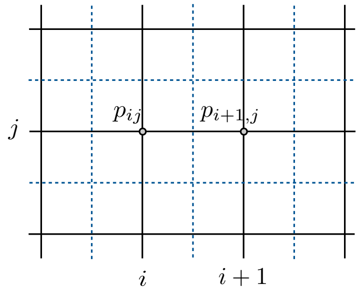
- leading to the Poisson equation:

$$\boxed{\frac{p_{i+1} - 2p_i + p_{i-1}}{\Delta x^2} = \frac{\rho}{\Delta t} = \frac{u_{i+1/2}^* - u*_{i-1/2}}{\Delta x}}$$

This completely eliminates the problem of small-scale oscillations!

Navier-Stokes **in 2D**

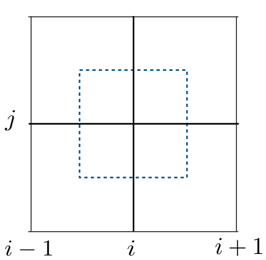
- 2D staggered grid

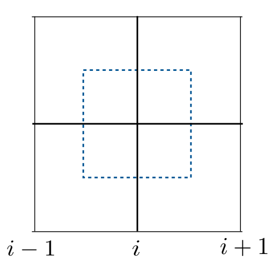


- Write "staggered-grid" finite difference approximations for:

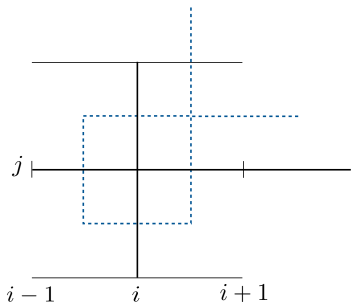
$$\frac{\partial u}{\partial t} \rightarrow$$
$$\frac{\partial u^2}{\partial t} \rightarrow$$
$$\frac{\partial uv}{\partial y} \rightarrow$$
$$\frac{\partial p}{\partial x} \rightarrow$$
$$\frac{\partial^2 u}{\partial x^2} \rightarrow$$
$$\frac{\partial^2 u}{\partial y^2} \rightarrow$$

- horizontal velocity located at the mid-points of the vertical "cell edges" and vertical velocity at the midpoints of the horizontal cell edges.





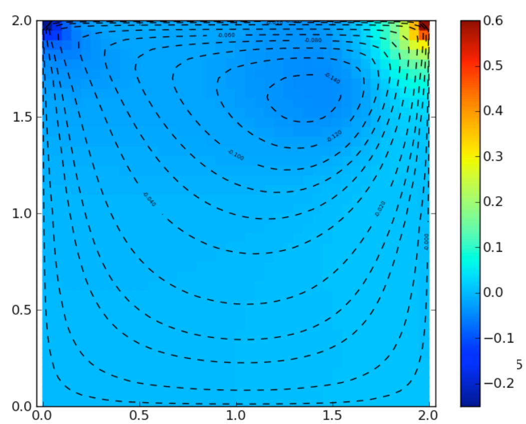
- to get, e.g., convection terms



Navier-Stokes **Collocated vs. Staggered grid**

- Cavity flow:

Step 11
41x41 mesh
dx=dy=0.05
nu=0.01 giving
Re=200



historical notes

- HW'65 introduced "marker and cell method"
- included a ste of marker particles to follow free surfaces (now obsolete)
- current usage of "MAC method"
- *projection method using a staggered grid*

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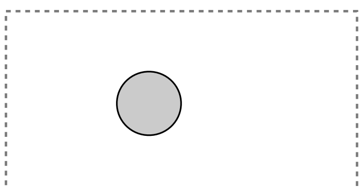
Navier-Stokes **Boundary Conditions**

$$\nabla \cdot \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \vec{u}$$

- in practice, B.C.s are stipulated separately for the different types:



Applied Mechanics Reviews MAY 2006, Vol. 59 / 107

On Boundary Conditions for Incompressible Navier-Stokes Problems

Dietmar Rempfer
Department of Materials, Mechanical and Aerospace Engineering,
Illinois Institute of Technology,
Chicago, IL 60616

We revisit the issue of finding proper boundary conditions for the field equations describing incompressible flow problems, for quantities like pressure or vorticity, which often do not have immediately obvious "physical" boundary conditions. Most of the issues are discussed for the example of a primitive-variables formulation of the incompressible Navier-Stokes equations in the form of momentum equations plus the pressure Poisson equation. However, analogous problems also exist in other formulations, some of which are briefly reviewed as well. This review article cites 95 references.
[DOI: 10.1115/1.2177683]

Keywords: flow simulation, numerical methods, Navier-Stokes equations

- “fractional step method” or “projection method”

- the projector is not identical with the pressure

JOURNAL OF COMPUTATIONAL PHYSICS **59**, 308–323 (1985)

Application of a Fractional-Step Method to Incompressible Navier–Stokes Equations

J. KIM AND P. MOIN

Computational Fluid Dynamics Branch, NASA Ames Research Center, Moffett Field, California 94035

Received March 15, 1984; revised September 4, 1984

$$\frac{u_i^{n+1} - \hat{u}_i}{\Delta t} = -G(\phi^{n+1}),$$

overall accuracy of the splitting method is still second order. Note that ϕ is different from the original pressure: in fact, $p = \phi + (\Delta t/2\,\mathrm{Re})\nabla^2\phi$. All the spatial derivatives

- equation for $\tilde{p}$

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The exploration of this problem of finding appropriate boundary conditions for the pressure forms the core of the present paper. As mentioned in our introduction, in addressing this question we are dealing with a highly controversial issue. In order to do so,

INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 7, 1111–1145 (1987)

ON PRESSURE BOUNDARY CONDITIONS FOR THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS*

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AND

ROBERT L. SANI

CIRES and Department of Chemical Engineering, University of Colorado, Boulder, Colorado 80309, U.S.A.

$$(\partial \mathbf{u}/\partial t) + \nabla P = \nu \nabla^2 \mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} \equiv \mathbf{f}$$

and

$$\nabla\cdot(\partial \mathbf{u}/\partial t) = 0 \quad \text{in} \quad \Omega;$$

these imply

$$\nabla^2 P = \nabla\cdot\mathbf{f} \quad \text{in} \quad \Omega;$$

also

$$\partial P/\partial n = \mathbf{n}\cdot(\mathbf{f} - (\partial \mathbf{u}/\partial t)) \quad \text{on} \quad \Gamma.$$

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cal example below, it is illegal to write down the momentum equation (1) taken at the boundary and derive a pressure boundary condition from it by simply projecting the result on the wall-normal coordinate η.

Despite the progress in understanding that had been achieved by the contributions mentioned above, in 1987 Gresho and Sani perpetuated and greatly contributed to the confusion in this area

situation any more settled now? Unfortunately, the answer to that question is negative: Improper pressure boundary conditions were still presented in review articles in the 1990s [12,60], and they found their way into some of the newest textbooks on computational fluid mechanics [13,61].

- Final note …

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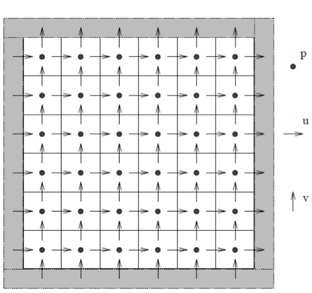
equations are evaluated at velocity nodes, and the continuity equation is enforced for each cell. One important advantage of using the staggered mesh for incompressible flows is that ad hoc pressure boundary conditions are not required



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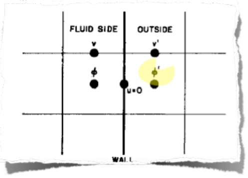
wall as shown in Fig. 4. The vertical velocities are simply reversed across the wall. Since $D = 0$ in the fluid cell, it follows that the vanishing of D' is accomplished only if $u' = +u_1$, in contrast to the requirement for u' for a free-slip wall.

In summary: (a) for a free-slip wall normal velocity reverses while tangential velocity remains the same; (b) for a no-slip wall normal velocity remains the same, while tangential velocity reverses.

For case (a) the pressure condition has been derived and has a simple form. For case (b) the no-slip wall, if vertical

$$\varphi' = \varphi_1 \pm g_x\,\delta x \pm (2\nu u_1/\delta x). \qquad (11)$$



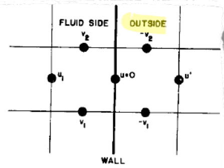


Fig. 4. Reflection of field variables near a wall.