Problem 1

(a) Let's just start by counting. $P(X = 1)$ must be 0, since if 1 is chosen, the other ball must be larger than 1. So $f_X(1) = 0$.

What about 2? For 2 to be the larger of the balls, the other must be 1. So we have to draw 1 and 2. That's the only draw allowed, out of the $\binom{5}{2}$ possible draws. So $f_X(2) = \frac{1}{10}$.

Now, what about general k? For k to be the larger of the balls, the other must be $k - 1, k - 2, ..., 1$. So there are $k - 1$ draws allowed, out of $\binom{5}{2}$ possible draws. So $f_X(k) = \frac{k-1}{10}$.

(b) What are the possible values of V? The two smallest numbers are 1 and 2, so 3 is V's smallest value. Similarly, V can't be greater than 9. So, we'll just go and exhaustively list all the possibilities.

How can we get a 3? Drawing 1 and 2 is the only way, clearly. So of the 10 possible draws, only 1 draw gives us 3, making $f_V(3) = \frac{1}{10}$.

How can we get a 4? Only by drawing 1 and 3; 2 and 2 isn't a possible draw, since there's only one ball with each number. So $f_V(4) = \frac{1}{10}$.

How can we get a 5? By drawing 1 and 4, or by drawing 2 and 3. So $f_V(5) = \frac{2}{10}$.

How can we get a 6? By drawing 1 and 5, or by drawing 2 and 4. So $f_V(6) = \frac{2}{10}$.

How can we get a 7? By drawing 2 and 5, or by drawing 3 and 4. So $f_V(7) = \frac{2}{10}$.

How can we get an 8? By drawing 3 and 5. $f_V(8) = \frac{1}{10}$.

How can we get a 9? By drawing 4 and 5. $f_V(9) = \frac{1}{10}$.

Problem 2

We have the formula $f_X(k) = F_X(k) - F_X(k-1)$. Using it will greatly simplify the calculations for this problem. What is $F_X(k)$? It's the probability that all three of the dice are k or lower. By the Multiplication Principle, the probability of that occuring is simply $\frac{k^3}{6^3}$. Hence, $f_X(k) = \frac{k^3-(k-1)^2}{6^3}$, except for k=1, when the answer is just $\frac{1}{6^3}$.

Problem 3

We'll list the 1,2,2,3,3,4 die first and the other second. The sample space is simply the set of ordered pairs { (a,b) | a = 1,2,3,4 and b = 1,3,4,5,6,8 }.

If we want to show the probabilities are the same as for an ordinary pair dice, we just have to go through and count. There's obviously 36 different ways the dice could land, but since numbers on the first die are repeated, some of the 36 configurations give the same "roll". We have to be careful while we count.

How can we get 2? Just (1,1). f(2) = P(1,1) = 1/36.

How can we get 3? (2,1), which can happen two different ways, so f(3) = P(2,1) = 2/36.

How can we get 4? (1,3) and (3,1). (3,1) can happen two different ways. So f(4) = 3/36.

How can we get 5? (1,4) and (2,3) and (4,1). (2,3) can happen two different ways. That's 4/36.

How can we get 6? (1,5) and (2,4) and (3,3). (2,4) and (3,3) can both happen two different ways. So 5/36.

I'll keep writing it out in simplified notation:

7: (1,6), (2,5), (2,5), (3,4), (3,4), (4,3). 6/36

8: (2,6), (2,6), (3,5), (3,5), (4,4) . 5/36

9: (1,8), (3,6), (3,6), (4,5). 4/36

10: (2,8), (2,8), (4,6). 3/36

11: (3,8), (3,8). 2/36

12: (4,8). 1/36

Problem 4

Method 1: As the cdf $F_Y(y) = P(Y \leq y)$, which implies:

$$\begin{aligned} P\Big(\frac{1}{2} < Y \leq \frac{3}{4}\Big) &= P\Big(Y \leq \frac{3}{4}\Big) - P\Big(Y \leq \frac{1}{2}\Big) \\ &= F_Y\Big(\frac{3}{4}\Big) - F_Y\Big(\frac{1}{2}\Big) \\ &= \Big(\frac{3}{4}\Big)^2 - \Big(\frac{1}{2}\Big)^2 \\ &= \frac{5}{16} \end{aligned}$$

Method 2: We first to calculate pdf by taking derivative of cdf

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ 2y & 0 \leq y < 1 \\ 0 & y \geq 1 \end{cases}$$

Because Y is continuous random variable, that means $P(Y = a) = 0$, where a

is any fixed constant. Then, we have:

$$P\Big(\frac{1}{2} < Y \le \frac{3}{4}\Big) = \int_{1/2}^{3/4} f_Y(y)dy$$
$$= \frac{5}{16}$$

Problem 5

(a) As $e = 2.718$ and Y is obviously continuous random variable, that means $P(Y = a) = 0$, where a is any fixed constant. Then, we have:

$$P(Y < 2) = F_Y(2) = \ln 2$$

(b)
$$P\Big(2 < Y \le 2\frac{1}{2}\Big) = F_Y\Big(2\frac{1}{2}\Big) - F_Y(2) = \ln\frac{5}{2} - \ln 2 = \ln\frac{5}{4}$$

(c)
$$P\Big(2 < Y < 2\frac{1}{2}\Big) = P\Big(2 < Y \le 2\frac{1}{2}\Big) - P\Big(Y = \frac{1}{2}\Big) = \ln\frac{5}{4}$$

(d)

$$f_Y(y) = \begin{cases} 0 & y < 1 \\ 1/y & 1 \le y \le e \\ 0 & e < y \end{cases}$$

Problem 6

By Definition 3.4.3, and as $y \ge 0$, we have:

$$\begin{aligned} F_Y(y) &= \int_0^y te^{-t}dt \\ &= -\int_0^y tde^{-t} \\ &= -\Big(te^{-t}\Big|_0^y - \int_0^y e^{-t}dt\Big) \\ &= -\Big(ye^{-y} + e^{-y} - 1\Big) \\ &= 1 - (1+y)e^{-y} \end{aligned}$$

`Problem 7`

First we try to find cdf of w. For $0 \le w \le 2$:

$$\begin{aligned} F_W(w) &= P(W \le w) = P(2Y \le w) \\ &= P\Big(Y \le \frac{w}{2}\Big) = F_Y\Big(\frac{w}{2}\Big) \\ &= \int_0^{\frac{w}{2}} 4y^3 dy \\ &= \frac{w^4}{16} \end{aligned}$$

Obviously, for $w < 0, F_W(w) = 0$ and for $w > 2, F_W(w) = 1$. Thus,

$$f_W(w) = \begin{cases} 0 & w < 0 \quad \text{or} \quad w > 2 \\ \frac{w^3}{4} & 0 \le w \le 2 \end{cases}$$

`Problem 8`

(a)

$$E(Y) = \int_0^1 y \cdot 3(1-y)^2 dy = \left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right]\Bigg|_0^1 = 3\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{4}.$$

(b)

$$E(Y) = \int_0^\infty y \cdot 4ye^{-2y} dy$$

Use u-v substitution twice with $u = y^2$, then y, and $v = e^{-2y}$ to get,

$$E(Y) = \int_0^\infty y \cdot 4ye^{-2y} dy = 1.$$

(c)

$$E(Y) = \int_{-\infty}^\infty y f_Y(y) dy = \int_0^1 \frac{3}{4} y dy + \int_2^3 \frac{1}{4} y dy =$$
$$\frac{3y^2}{8}\Bigg|_0^1 + \frac{y^2}{8}\Bigg|_2^3 = \frac{3}{8} + \frac{9}{8} - \frac{4}{8}.$$

(d)

$$E(Y) = \int_0^{\pi/2} y \sin(y) dy = \sin(y) - y\cos y|_0^{\pi/2} = (1-0) - (0-0) = 1.$$

`Problem 9`

Let the random variable be denoted by Y. As $Y > 0$,

$$
\begin{aligned}
EY &= \int_0^\infty y\lambda e^{-\lambda y} dy \\
&= -\Big(\int_0^\infty y de^{-\lambda y} \Big) \\
&= -\Big(ye^{-\lambda y}\Big|_0^\infty - \int_0^\infty e^{-\lambda y} dy \Big) \\
&= \frac{1}{\lambda}
\end{aligned}
$$

`Problem 10`

We need f_X before we can find E(X). What is the probability of all the faces being the same? Let's count how many ways that could happen. The first die can be any number, but then the others have to be the same. So 6 x 1 x 1. That's out of 6^3 possible rolls. Hence, $f_X(1) = \frac{6}{6^3}$.

What about all three being different? The first die can be any number, then the second can be any but the previous number, then the third can be any but the previous two. So 6 x 5 x 4 ways. Hence, $f_X(3) = \frac{6\times5\times4}{6^3}$.

Now, we're left with two different numbers. There's 6 x 5 different choices for what two numbers appear, and 3 different choices for how to order them (ie, aab, aba, baa). So 6 x 5 x 3 ways it could happen. Hence, $f_X(2) = \frac{6\times5\times3}{6^3}$.

Now that we have f_X, we can compute E(X) $= 1\times\frac{6}{6^3} + 2\times\frac{6\times5\times3}{6^3} + 3\times\frac{6\times5\times4}{6^3}$.

`Problem 11`

To calculate $E[W]$ we use Theorem 3.5.3

$$E[W] = E\left[(Y - \frac{2}{3})^2\right] = \int_0^1 \left(y - \frac{2}{3}\right)^2 \cdot 2y dy = 2\left[\frac{y^4}{4} - \frac{4y^3}{9} + \frac{2y^2}{9}\right]\Bigg|_0^1 = \frac{1}{18}.$$

`Problem 12`

To find the variance, we need $E(Y)$ and $E(Y^2)$.

$E(Y) = \int_0^1 y f_Y(y) = \int_0^1 y3(1-y)^2 = 3\int_0^1 y(1-2y+y^2) = 3\int_0^1 y - 2y^2 + y^3 = 3(\frac{1}{2} - \frac{2}{3} + \frac{1}{4})$

$E(Y^2) = \int_0^1 y^2 f_Y(y) = \int_0^1 y^2 3(1-y)^2 = 3\int_0^1 y^2(1-2y+y^2) = 3\int_0^1 y^2 - 2y^3 + y^4 = 3(\frac{1}{3} - \frac{2}{4} + \frac{1}{5})$

So, var(Y) $= 3(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}) - (3(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}))^2$.