Problem 1

So the probability of any single sighting is genuine is $p = \frac{1}{100000}$. To make this readable, we denote the number of sighting is genuine by random variable X. And obviously, X follows binomial distribution, which implies:

$$P(X = k) = \binom{10000}{k} p^k (1-p)^{10000-k}$$

And the event at least one is genuine equal to the complement of event none is genuine.

$$\begin{aligned} P(\text{at least 1 sighting is genuine}) &= 1 - P(\text{none is genuine}) \\ &= 1 - P(X = 0) \\ &= 1 - \binom{10000}{0} p^0 (1-p)^1 0000 \\ &= 1 - \left(\frac{99999}{100000}\right)^{10000} \\ &\approx 0.0952 \end{aligned}$$

Problem 2

$$P(a) = \binom{7}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^4 = \frac{35}{128}$$

And

$$\begin{aligned} P(b) &= \binom{7}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^7 + \binom{7}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^6 + \binom{7}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^5 \\ &= \frac{12393}{16348} \end{aligned}$$

That means b will more likely happen.

Problem 3

(a) Actually, we can see these crashes as a binomial distribution, because the number of flights in this country should be a known fix number per year, and the probability of each plane crash has the same probability.
The number of flights each year must be a very large number, and the probability of crashing is quite small. As theorem 4.2.1, when the $n \to \infty$ and $p \to 0$, this distribution is going to poisson distribution. Also as the number of crashed plane is integer, this is a discrete variable, which makes the assumption reasonable.

(b) To consider 4 or more, we can just think about its complement, that is 0,1,2 and 3 crashes in the next year. Denote the variable by X.

$$\begin{aligned} P(X = 4 \quad \text{or more}) &= 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3) \\ &= 1 - \frac{e^{-2.5}2.5^0}{0!} - \frac{e^{2.5}2.5^1}{1!} - \frac{e^{2.5}2.5^2}{2!} - \frac{e^{2.5}2.5^3}{3!} \end{aligned}$$

(c) In this part, the probability of two crashes in 3 month is equal to the probability of the time between two crashes is less than 3 months ($\frac{1}{4}$ year), which actually follows exponential distribution.

$$P(X < 3) = \int_0^{\frac{1}{4}} 2.5e^{-2.5y}dy = -e^{2.5y}\Big|_0^{\frac{1}{4}} = 1 - e^{-0.625}$$

Problem 4

(a) Yes, the binomial distribution would work. Consider each trial to be person chosen at random, and consider the trial to be a success if the student consumed alcoholic beverages.

(b) If $X \sim Bin(10, 0.7)$, the probability that exactly six people have consumed alcohol would be $f_X(6) = \binom{10}{6}(0.7)^6(0.3)^4$.

(c) Same as before! The probability of exactly four failures is the same as the probability of exactly six successes, if the total number of trials is ten.

(d) Now we have $X \sim Bin(5, 0.7)$. We want $P(X \leq 2)$. That's just $f_X(0) + f_X(1) + f_X(2) = \binom{5}{0}(0.7)^0(0.3)^5 + \binom{5}{1}(0.7)^1(0.3)^4 + \binom{5}{2}(0.7)^2(0.3)^3$.

(e) We still have $X \sim Bin(5, 0.7)$. Now we want $P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{5}{0}(0.7)^0(0.3)^5$.

Problem 5

(a) $X \sim Bin(3, 0.25)$. We want $P(X \geq 1)$. That's just $1 - f_X(0) = 1 - \binom{3}{0}(0.25)^0(0.75)^3$.

(b) $P(X = 2) = f_X(2) = \binom{3}{2}(0.25)^2(0.75)^1$.

(c) We still have $X \sim Bin(3, 0.25)$, but now X counts the number of heis, instead of nuns. $P(X = 1) = \binom{3}{1}(0.25)^1(0.75)^2$.

(d) Again, we still have $X \sim Bin(3, 0.25)$, but this time X is counting the number of gimels. $P(X \leq 2) = 1 - f_X(3) = 1 - \binom{3}{3}(0.25)^3(0.75)^0$.

Note: I was being lazy. I should have said something like: "Let X be the number of nuns after 3 rolls of the dreidel. Let Y be the number of heis after 3 rolls of the dreidel. Let Z be the number of gimels after 3 rolls of the dreidel. All three of them are described by the binomial distribution Bin(3, 0.25)." The point is that there are three different random variables; they all just happen to be identically distributed.

Problem 6

(a) Let X be the number of boys that the couple has. Assuming independence, this is a binomial random variable with success probability $p = 0.51$ and $n = 3$ trials. Thus

$$P(X = 2) = \binom{3}{2}(0.51)^2(0.49)^{3-2}.$$

(b) Let B and G represent having a boy and a girl respectively, and order the events chronologically. Then the possibilities for having two boys are GBB, BGB, BBG. The probabilities of each of these are (0.49)(0.51)(0.51) for GBB, (0.51)(0.49)(0.51) for BGB, and (0.51)(0.51)(0.49) for BBG. Since they are disjoint events, the probability that one of them occurs is the sum of the three, which is the same as the above answer.

(c) Using the approach from part (a) would be the same difficulty, but using the approach from part (b) would be difficult because instead of $\binom{3}{2} = 3$ possibilities, there would be $\binom{8}{3} = 56$ possibilities. We do not want to write all of these down!