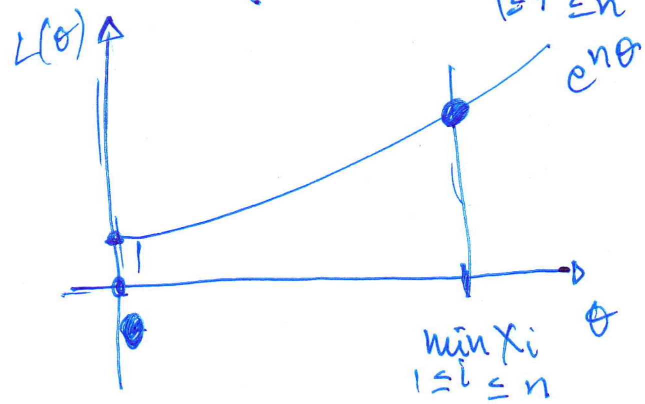
W7: WEDNESDAY :

Reading: Ch 5.2 & 5.4 LAM.

EX: $(X_1, \dots, X_n)$ - sample from density $f_X(x;\theta) = e^{-(x-\theta)}$

for $x \geq \theta$ and $\theta > 0$.

$$L(\theta) = e^{-\sum_{i=1}^{n} x_i} \cdot e^{n\theta} \cdot \mathbb{1}\left(0 < \theta < \min_{1 \le i \le n} x_i\right).$$



$\hat{\theta} = \min_{1 \le i \le n} X_i$ - rand. v. (never of θ)

(1) Distribution of $\hat{\theta}$

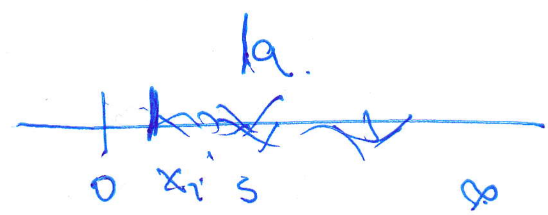
(2) $\mathbb{E}(\hat{\theta})$

(3) $\text{var}(\hat{\theta})$

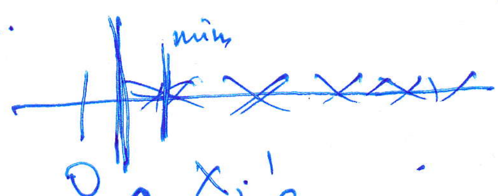
$\Big\}$ functions of θ

$\mathbb{P}(\hat{\theta} \le a) = \mathbb{P}(\min_{1 \le i \le n} X_i \le a)$

too hard.



$\mathbb{P}(\hat{\theta} \geq a) = \mathbb{P}(\min_{1 \le i \le n} X_i \geq a)$



$= \mathbb{P}(\underline{\text{all } X_i} \geq a)$

$= \mathbb{P}(\{X_1 \geq a\} \cap \{X_2 \geq a\} \cap \dots \cap \{X_n \geq a\}).$

$$\mathbb{P}(\hat\theta \geq a) = \prod_{i=1}^{n} \underbrace{\mathbb{P}(X_i \geq a)}$$

X_i - iid $\quad f_{X_i}(x;\theta) = e^{-(x-\theta)}, \ x \geq \theta, \ \theta > 0$

$$\mathbb{P}(X_i \geq a) = \int_a^\infty e^{-(x-\theta)}\,dx = \int_a^{+\infty} e^{-x}\cdot e^{\theta}\,dx =$$

$$= e^{\theta}\int_a^\infty e^{-x}\,dx = e^{\theta}\cdot\left(-e^{-x}\Big|_a^\infty\right)$$

$$= e^{\theta}\cdot\left(\cancel{-e^{-\infty}} + e^{-a}\right)$$

$\to 0$ ✓

If $a \leq \theta$: $\mathbb{P}(X_i \geq \theta) = 1$

Remember: $e^{-\infty} = 0$

$+\infty \cdot e^{-\infty} = 0$

$-\infty \cdot e^{-\infty} = 0$

$\cancel{\boxed{e^{+\infty} = +\infty}}$

$$\mathbb{P}(X_i \geq a) = e^{\theta - a} \quad \text{for } a \geq \theta.$$

$$\mathbb{P}(\hat\theta \geq a) = \prod_{i=1}^{n} \mathbb{P}(X_i \geq a) = \prod_{i=1}^{n} e^{\theta - a} = e^{n(\theta - a)}$$

(1) $\mathbb{P}(\hat\theta \leq a) = 1 - \mathbb{P}(\hat\theta \geq a) = 1 - e^{n(\theta - a)}$ } - cdf $\hat\theta$.

(2). $\mathbb{E}\hat\theta = ?$

Def An estimator is unbiased. If $\mathbb{E}(\hat\theta) = \theta$.

$\mathbb{E}\hat\theta = ?$ - Find density of $\hat\theta$

- compute $\mathbb{E}$.

$$f_{\hat{\theta}}(a) = \frac{d}{da}\{P(\hat{\theta} \leq a)\} = \frac{d}{da}\left(1 - e^{n\theta - na}\right).$$

$$= -\left(\frac{d}{da} e^{n\theta - na}\right) = -e^{n\theta - na} \cdot \left(\frac{d}{da}(n\theta - na)\right).$$

$$= -e^{n\theta - na} \cdot (-n) = n \cdot e^{n\theta - na}.$$

if $a \geq \theta$..

$$E\hat{\theta} = \int_{-\infty}^{+\infty} a \cdot f_{\hat{\theta}}(a)\,da = \int^{\infty} a \cdot n \cdot e^{n\theta - na}\,da.$$

$$= n \cdot e^{n\theta} \cdot \underbrace{\int_{\theta}^{\infty} a \cdot e^{-na}\,da.}$$

$$u = a \qquad dv = e^{-na}\,da.$$

$$du = da \qquad v = \int e^{-na}\,da = -\frac{1}{n} \cdot e^{-na}.$$

$$\int_{\square}^{\square} u\,dv = u \cdot v \Big|_{\square}^{\square} - \int_{\square}^{\square} v\,du. \quad \text{FORMULA.}$$

$$E\hat{\theta} = n \cdot e^{n\theta} \left(a \cdot \left(-\frac{1}{n}\right) \cdot e^{-na} \Big|_{\theta}^{\infty} - \int_{\theta}^{\infty} \left(-\frac{1}{n} e^{-na}\right) da \right)$$

$$E\hat{\theta} = n \cdot e^{n\theta} \left(0 - \frac{(-\theta)}{n} \cdot e^{-n\theta} + \int_{\theta}^{\infty} \frac{1}{n} e^{-na}\,da \right).$$

$$= n \cdot e^{n\theta} \left(\frac{\theta}{n} \cdot e^{-n\theta} + \frac{1}{n} \cdot \left(-e^{-na} \frac{1}{n}\right) \Big|_{\theta}^{\infty} \right).$$

$$= n e^{n\theta} \left(\frac{\theta e^{-n\theta}}{n} + \frac{1}{n^2} \left(e^{-n\theta}\right) \right).$$

EX: $(X_1, \dots, X_n)$ to be a sample from density.

$$f_X(x;\theta) = \frac{2}{\theta^2} \quad \text{if} \quad 0 \le x \le \theta.$$

ARE $\hat{\theta}_1 = \frac{3}{2}\left(\sum_{i=1}^{n} X_i\right)\cdot\frac{1}{n}$ and

$$\hat{\theta}_2 = \left(\frac{2n+1}{2n}\right)\cdot X_{max} \qquad \left(X_{max} = \max_{1\le i\le n} X_i\right).$$

both unbiased? Which one is more efficient?

Def: If $\hat{\theta}_1$ and $\hat{\theta}_2$ are both unbiased estimators then $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$

if $\mathrm{var}(\hat{\theta}_1) \le \mathrm{var}(\hat{\theta}_2)$.

$$\hat{\theta}_1 = \frac{3}{2n}\left(\sum_{i=1}^{n} X_i\right).$$

$$E\,\hat{\theta}_1 = E\left(\frac{3}{2n}\sum_{i=1}^{n} X_i\right) = \frac{3}{2n}\cdot E\left(\sum_{i=1}^{n} X_i\right) = \frac{3}{2n}\sum_{i=1}^{n} E X_i$$

RULE. For any $X_1, \dots, X_n$.

$$E(X_1 + \dots + X_n) = E(X_1) + \dots + E(X_n)$$

$$E X_i = \int_{-\infty}^{+\infty} x \cdot f(x;\theta)\,dx = \int_0^{\theta} x \cdot \frac{2}{\theta^2}\cdot dx$$

$$= \frac{2}{\theta^2}\int_0^{\theta} x\,dx = \frac{2}{\theta^2}\cdot\left(\frac{1}{2}x^2\Big|_0^{\theta}\right)$$

$$= \frac{2}{\theta^2}\left(\frac{1}{2}\theta^2 - 0\right) = \dots$$