

WT: FRIDAY:

READING: 5.2, 5.4 LAM
5.5.
5.3

METHOD ~~OF~~ MOMENTS

MIDTERM II

REVIEW

TUESDAY, 7-8 pm

ROOM: ~~TBA~~

PETERSON 103

$\Theta X: X_1, \dots, X_n \sim f_X(x; \theta) = \frac{2x}{\theta^2}$, for $0 \leq x \leq \theta$.

ARE $\hat{\theta}_1 = \frac{3}{2} \bar{X}$ and $\hat{\theta}_2 = \frac{2n+1}{2n} X_{\max}$ BOTH UNBIASED EST?

WHICH ONE IS MORE EFFICIENT?

Def: AN ESTIMATOR IS UNBIASED IFF $E[\hat{\theta}] = \theta$.

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

$$X_{\max} = \max_{1 \leq i \leq n} X_i$$

$$\hat{\theta}_1 = \frac{3}{2n} \sum_{i=1}^n X_i$$

$$\begin{aligned} E[\hat{\theta}_1] &= E\left(\frac{3}{2n} \sum_{i=1}^n X_i\right) = \frac{3}{2n} E\left(\sum_{i=1}^n X_i\right) = \frac{3}{2n} \sum_{i=1}^n E(X_i) \\ &= \frac{3}{2n} \sum_{i=1}^n \left(\frac{2}{3} \theta\right) = \frac{3}{2n} \cdot n \cdot \frac{2}{3} \cdot \theta = \theta \end{aligned}$$

$$\text{RULE: } E\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n E X_i \text{ FOR ANY } X_1, \dots, X_n.$$

SINCE $X_1, \dots, X_n$ FORM A SAMPLE

$X_1, \dots, X_n$

INDEPENDENT R.V.

IDENTICALLY DISTRIBUTED

WITH $f_X(x; \theta)$

$$X_1 \sim f_X(x; \theta) = \frac{2x}{\theta^2} \text{ if } 0 \leq x \leq \theta$$

$$E X_1 = ?$$

$$\begin{aligned} E X_1 &= \int_0^{\theta} x \cdot f_X(x; \theta) dx = \int_0^{\theta} x \cdot \frac{2x}{\theta^2} dx = \frac{2}{\theta^2} \int_0^{\theta} x^2 dx = \frac{2}{\theta^2} \left[\frac{x^3}{3} \right]_0^{\theta} \\ &= \frac{2}{\theta^2} \cdot \frac{\theta^3}{3} = \frac{2}{3} \cdot \theta \end{aligned}$$

DISTRIBUTION OF $\hat{\theta}_2 = ?$

(1) $P(\hat{\theta}_2 \leq a)$

$\hat{\theta}_2 = \frac{2n+1}{2n} \cdot \{X_{\max}\}$ (0, \theta)

$a=1$ - never happens.

~~$a \in (0, \infty)$~~

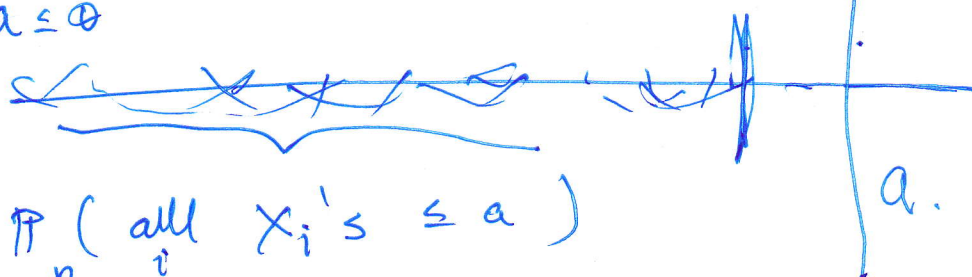
$a \in (0, \frac{2n+1}{2n} \cdot \theta)$

~~$a = (0, \infty)$ - never true.~~

~~$a \in (0, \infty)$~~

(2) $P(X_{\max} \leq a) = P(\max_{1 \leq i \leq n} X_i \leq a)$

$0 \leq a \leq \theta$



$= P(\text{all } X_i \text{'s} \leq a)$

$= \prod_{i=1}^n P(X_i \leq a)$

X_i 's ind.

$P(X_1 \leq a) = \int_0^a x \cdot \frac{2x}{\theta^2} dx$

$f_{X_1}(x; \theta) = \frac{2x}{\theta^2}, 0 < x < \theta$

$= \int_0^a \frac{2x}{\theta^2} dx$

$= \frac{2}{\theta^2} \int_0^a x dx = \frac{2}{\theta^2} \cdot \frac{1}{2} x^2 \Big|_0^a = \frac{a^2}{\theta^2}$

$P(X_{\max} \leq a) = \prod_{i=1}^n P(X_i \leq a)$

$= \frac{a^{2n}}{\theta^{2n}}$

- dist. of $X_{\max}$.

$P(\frac{2n+1}{2n} \hat{\theta}_2 \leq b) = P(X_{\max} \leq \frac{2n \cdot b}{2n+1}) \dots$

$$E X_{\max} = ?$$

$$E \hat{\theta}_2 = E \left(\frac{2n+1}{2n} X_{\max} \right) \\ = \frac{2n+1}{2n} E(X_{\max}),$$

$E X_{\max} = ?$ ← need density of $X_{\max}$.

$$f_{X_{\max}}(a) = \frac{d}{da} P(X_{\max} \leq a) = \frac{d}{da} \left(\frac{a^{2n}}{\theta^{2n}} \right) = \frac{1}{\theta^{2n}} \cdot 2n a^{2n-1}$$

$$= \frac{1}{\theta^{2n}} \cdot 2n \cdot a^{2n-1}$$

$$0 < a < \theta \quad \checkmark \quad \int_0^a f_{X_{\max}}(a) da.$$

$$E X_{\max} = \int_0^{\theta} X \cdot \left(\frac{1}{\theta^{2n}} 2n \cdot a^{2n-1} \right) da.$$

$$= \frac{2n}{\theta^{2n}} \int_0^{\theta} a^{2n} da = \frac{2n}{\theta^{2n}} \cdot \frac{1}{2n+1} a^{2n+1} \Big|_0^{\theta} = \frac{2n}{\theta^{2n}} \cdot \frac{\theta^{2n+1}}{2n+1}$$

$$= \frac{2n}{\theta^{2n}} \cdot \frac{1}{2n+1} \cdot a^{2n+1} \Big|_0^{\theta} = \frac{2n}{2n+1} \cdot \frac{\theta^{2n+1}}{\theta^{2n}}$$

$$= \frac{2n}{2n+1} \cdot \theta.$$

$$E \hat{\theta}_2 = E \left(\frac{2n+1}{2n} X_{\max} \right) = \frac{2n+1}{2n} E X_{\max} = \frac{2n+1}{2n} \cdot \frac{2n}{2n+1} \cdot \theta = \theta$$

Def: If $\hat{\theta}_1$ and $\hat{\theta}_2$ are both unbiased est. of θ .
Then $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$
if $\text{var } \hat{\theta}_1 < \text{var } \hat{\theta}_2$

$$\hat{\theta}_1 = \frac{3}{2} \bar{X} = \frac{3}{2} \frac{1}{n} \sum_{i=1}^n X_i$$

$$\hat{\theta}_2 = \frac{2n+1}{2n} X_{\max}$$

$$\text{var}(\hat{\theta}_1) = \text{var}\left(\frac{3}{2n} \sum_{i=1}^n X_i\right) = \frac{9}{4n^2} \text{var}\left(\sum_{i=1}^n X_i\right)$$

RULE: $\text{var}(a \cdot X) = a^2 \text{var}(X)$

$$\text{var}(\hat{\theta}_1) = \frac{9}{4n^2} \sum_{i=1}^n \text{var}(X_i) = \frac{9}{4n^2} \cdot n \cdot \frac{1}{18} \cdot \theta^2 = \frac{\theta^2}{8n}$$

RULE: If $X_1, \dots, X_n$ are independent R.V.'s
 THEN $\text{var}(X_1 + \dots + X_n) = \text{var}(X_1) + \dots + \text{var}(X_n)$

$$X_i \sim f_X(x; \theta) = \frac{2x}{\theta^2}$$

$$0 < x < \theta$$

$$\#X_i = \frac{2}{3}\theta$$

$$\text{var } X_i = \#(X_i^2) - (\#X_i)^2$$

$$\begin{aligned} \#(X_i^2) &= \int_0^\theta x^2 \cdot f_X(x; \theta) dx = \int_0^\theta \frac{2x^3}{\theta^2} dx = \frac{2}{\theta^2} \cdot \frac{1}{4} x^4 \Big|_0^\theta \\ &= \frac{2}{\theta^2} \cdot \frac{1}{4} \cdot \theta^4 = \frac{\theta^2}{2} \end{aligned}$$

$$\text{var } X_i = \frac{\theta^2}{2} - \left(\frac{2}{3}\theta\right)^2 = \frac{\theta^2}{2} - \frac{4}{9}\theta^2 = \frac{1}{18}\theta^2$$

$$\text{var } \hat{\theta}_2 = ?$$

$$\hat{\theta}_2 = \frac{2n+1}{2n} X_{\max}$$

$$f_{X_{\max}}(a) = 2n \cdot \frac{a^{2n-1}}{\theta^{2n}}, \quad 0 < a < \theta$$

$$\text{var } \hat{\theta}_2 = E(\hat{\theta}_2^2) - (E(\hat{\theta}_2))^2$$

$$E \hat{\theta}_2 = \theta$$

$$E(\hat{\theta}_2^2) = ?$$

$$E(X_{\max}^2) = \int_0^\theta a^2 \cdot 2n \cdot \frac{a^{2n-1}}{\theta^{2n}} da = \frac{2n}{\theta^{2n}} \int_0^\theta a^{2n+1} da$$

$$= \frac{2n}{\theta^{2n}} \cdot \frac{1}{2n+2} \cdot a^{2n+2} \Big|_0^\theta = \frac{2n}{2n+2} \cdot \frac{\theta^{2n+2}}{\theta^{2n}}$$

$$= \frac{2n}{2n+2} \theta^2$$

$$E(\hat{\theta}_2^2) = E\left(\left(\frac{2n+1}{2n} X_{\max}\right)^2\right) = E\left(\frac{(2n+1)^2}{(2n)^2} X_{\max}^2\right)$$

$$= \frac{(2n+1)^2}{(2n)^2} E X_{\max}^2 = \frac{(2n+1)^2}{(2n)^2} \cdot \frac{2n}{2n+2} \cdot \theta^2$$

$$= \frac{(2n+1)^2}{2n \cdot (2n+2)} \theta^2$$

$$\text{var } \hat{\theta}_2 = \frac{(2n+1)^2}{2n \cdot (2n+2)} \theta^2 - \theta^2 = \frac{1}{4n(n+1)} \theta^2$$

$$\text{var } \hat{\theta}_2 < \text{var } \hat{\theta}_1$$

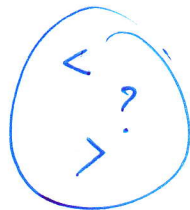
jbradic

Math 183 A09352495 Homework 6.pdf

[21/Feb/2013:12:32:55 -0800] - p5880c-238480 - math64.ucsd.edu

$$\text{var } \hat{\theta}_1$$

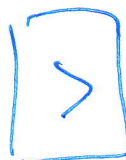
$$\frac{1}{8n} \theta^2$$



$$\cdot \text{var } \hat{\theta}_2$$

$$\frac{1}{4n(n+1)} \theta^2$$

$$\left(\right) \frac{1}{2}$$



$$\frac{1}{n+1} \left(\right)$$

.

$$\frac{1}{2} < \frac{1}{n+1} ?$$

$$n+1 < 2.$$

$$n < 1 ?$$

$\hat{\theta}_2$ is more efficient

Relative efficiency

$$\frac{\text{var}(\hat{\theta}_1)}{\text{var}(\hat{\theta}_2)} = \frac{\cancel{\theta^2} \cdot \cancel{4n(n+1)}}{\cancel{8n} \cdot \cancel{\theta^2}} = \frac{n+1}{2}$$

$$\frac{n+1}{2} > 1$$

$$n+1 > 2$$

$$\boxed{n > 1}$$

$$\text{var}(\hat{\theta}_1) > \text{var}(\hat{\theta}_2) \checkmark$$