W7: FRIDAY:

READING: 5.2, 5.4 L 7 14
5.5.
5.3
METHOD ~~OF~~ MOMENTS

MIDTERM II
REVIEW

TUESDAY, 7–8 pm
ROOM: ~~TBA~~.
PETERSON 103

EX: $X_1, \dots, X_n \sim f_X(x;\theta) = \frac{2x}{\theta^2}$, for $0 \le x \le \theta$

ARE $\hat{\theta}_1 = \frac{3}{2}\bar{X}$ and $\hat{\theta}_2 = \frac{2n+1}{2n} \cdot X_{max}$ BOTH UNBIASED EST?

WHICH ONE IS MORE EFFICIENT.

Def: AN ESTIMATOR IS unbiased iff $E[\hat{\theta}] = \theta$.

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ $\qquad X_{max} = \max_{1 \le i \le n} X_i$

$$\hat{\theta}_1 = \frac{3}{2n} \cdot \sum_{i=1}^{n} X_i$$

$$E\hat{\theta}_1 = ? = E\left(\frac{3}{2n}\sum_{i=1}^{n} X_i\right) = \frac{3}{2n} E\left(\sum_{i=1}^{n} X_i\right) = \frac{3}{2n}\sum_{i=1}^{n} E(X_i)$$

$$= \frac{3}{2n} \cdot \sum_{i=1}^{n} \left(\frac{2}{3}\theta\right) = \frac{\cancel{3}}{\cancel{2n}} \cdot \cancel{n} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \theta = \theta$$

$$\boxed{\text{RULE: } E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} EX_i \quad \text{FOR ANY } X_1, \dots, X_n.}$$

SINCE $X_1, \dots, X_n$ FORM A SAMPLE ← $X_1, \dots, X_n$
INDEPENDENT R.V.
IDENTICALLY DISTRIBUTED
WITH $f_X(x;\theta)$

$X_1 \sim f_X(x;\theta) = \frac{2x}{\theta^2}$ if $0 \le x \le \theta$

$EX_1 = ?$

$$EX_1 = \int_0^{\theta} x \cdot f_X(x;\theta)\,dx = \int_0^{\theta} x \cdot \frac{2x}{\theta^2}\,dx = \frac{2}{\theta^2}\int_0^{\theta} x^2\,dx = \frac{2}{\theta^2}\frac{x^3}{3}\Big|_0^{\theta}$$

$$= \frac{2}{\cancel{\theta^2}} \cdot \frac{\theta^{\cancel{3}}}{3} = \frac{2}{3}\theta$$

DISTRIBUTION OF $\hat{\theta}_2 = ?$

(1) $P(\hat{\theta}_2 \le a)$ $\qquad \hat{\theta}_2 = \frac{2n+1}{2n} \{X_{max}\}$. $\quad (0, \theta)$

$a = 1$ - never happens.

$\cancel{a = (0, \infty)}$

$a = (0, \theta)$ $\qquad \boxed{a \in \left(0, \frac{2n+1}{2n} \cdot \theta\right)}$.

$a = (0, x)$ - never ~~true~~.

$\cancel{a = (\theta, \infty)}$

(2) $P(X_{max} \le a) = P\left(\max_{1 \le i \le} X_i \le a\right)$

$0 \le a \le \theta$

a.

$= P(\text{all } X_i\text{'s} \le a)$

$= \prod_{i=1}^{n} P(X_i \le a)$

X_i's ind.

$P(X_1 \le a) = \cancel{\boxed{\int_0^\theta x \cdot \frac{2x}{\theta^2}\, dx}} = EX_1$ $\quad = \int_0^a \frac{2x}{\theta^2}\, dx$.

$f_{X_1}(x; \theta) = \frac{2x}{\theta^2}, \quad 0 < x < \theta$

$= \frac{2}{\theta^2} \int_0^a 2x\, dx = \frac{2}{\theta^2} \frac{1}{2} x^2 \Big|_0^a = \frac{a^2}{\theta^2}$

$P(X_{max} \le a) = \prod_{i=1}^{n} P(X_i \le a) = \frac{a^{2n}}{\theta^{2n}}$ - dist. of X_{max}.

$P\left(\underbrace{\frac{2n+1}{2n} X_{max}}_{\hat{\theta}_2} \le b\right) = P\left(X_{max} \le \frac{2n \cdot b}{2n+1}\right) \ldots$

$\mathbb{E} X_{max} = ?$ $\qquad \mathbb{E}\hat{\theta}_2 = \mathbb{E}\left(\frac{2n+1}{2n} X_{max}\right)$

$$= \frac{2n+1}{2n} \mathbb{E}(X_{max}),$$

$\mathbb{E} X_{max} = ?$ ⟵ need density of X_{max}.

$$f_{X_{max}}(a) = \frac{d}{da} \mathbb{P}(X_{max} \le a) = \frac{d}{da}\left(\frac{a^{2n}}{\theta^{2n}}\right) = \cancel{\frac{1}{\theta^{2n}} \cdot 2n\, a^{2n+1}}$$

$$= \frac{1}{\theta^{2n}} \cdot 2n \cdot a^{2n-1}$$

$0 < a < \theta$ ✓ $\qquad = \int a \cdot f_{X_{max}}(a)\, da.$

$$\mathbb{E} X_{max} = \int_0^\theta \cancel{X}\, a \cdot \frac{1}{\theta^{2n}} 2n \cdot a^{2n-1}\, da.$$

$$= \frac{2n}{\theta^{2n}} \int_0^\theta a^{2n}\, da. \cancel{= \frac{2n}{\theta^{2n}} \cdot \frac{1}{2n+1} a^{2n+1} \Big|_0^\theta = \frac{\theta^{2n+1}}{\theta^{2n}}}$$

$$= \frac{2n}{\theta^{2n}} \frac{1}{2n+1} \cdot a^{2n+1} \Big|_0^\theta = \frac{2n}{2n+1} \frac{\theta^{2n+1}}{\theta^{2n}}$$

$$= \frac{2n}{2n+1} \cdot \theta.$$

$$\mathbb{E}\hat{\theta}_2 = \mathbb{E}\left(\frac{2n+1}{2n} X_{max}\right) = \frac{2n+1}{2n} \mathbb{E} X_{max} = \cancel{\frac{2n+1}{2n}} \cdot \cancel{\frac{2n}{2n+1}} \cdot \theta$$

Def: If $\hat{\theta}_1$ and $\hat{\theta}_2$ are both unbiased est. of θ, then $\hat{\theta}_1$ is more efficient then $\hat{\theta}_2$ if $\operatorname{var} \hat{\theta}_1 < \operatorname{var} \hat{\theta}_2$

$$\hat{\theta}_1 = \frac{3}{2}\bar{X} = \frac{3}{2}\frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\hat{\theta}_2 = \frac{2n+1}{2n} X_{max}$$

$$var(\hat{\theta}_1) = var\left(\frac{3}{2n}\sum_{i=1}^{n} X_i\right) = \frac{9}{4n^2} var\left(\sum_{i=1}^{n} X_i\right)$$

RULE! $\boxed{var(a \cdot X) = a^2 var(X)}$

$$var(\hat{\theta}_1) = \frac{9}{4n^2}\sum_{i=1}^{n} var(X_i) = \cancel{\frac{9}{4n^2}} \cdot \cancel{n} \cdot \frac{1}{\cancel{18}_2} \cdot \theta^2 = \frac{\theta^2}{8n}$$

RULE! If $X_1, \ldots, X_n$ are <u>independent</u> R.V.'s THEN

$$var(X_1 + \ldots + X_n) = var(X_1) + \ldots + var(X_n)$$

$$X_i \sim f_X(x;\theta) = \frac{2x}{\theta^2} \qquad 0 < x < \theta.$$

$$E X_i = \frac{2}{3}\theta.$$

$$\boxed{var X_i = E(X_i^2) - (E X_i)^2}$$

$$E(X_i^2) = \int_0^{\theta} x^2 \cdot f_X(x;\theta)\, dx = \int_0^{\theta} \frac{2x^3}{\theta^2}\, dx = \frac{\cancel{2}}{\theta^2} \cdot \frac{1}{\cancel{4}_2} x^4 \Big|_0^{\theta}$$

$$= \frac{\cancel{2}1}{2\theta^2} \cdot \theta^4 = \frac{\theta^2}{2}.$$

$$var X_i = \frac{\theta^2}{2} - \left(\frac{2}{3}\theta\right)^2 = \frac{\theta^2}{2} - \frac{4}{9}\theta^2 = \frac{1}{18}\theta^2$$

$\text{var}\, \hat{\theta}_2 = ?$ $\qquad \hat{\theta}_2 = \frac{2n+1}{2n} X_{max}.$

$$f_{X_{max}}(a) = 2n \cdot \frac{a^{2n-1}}{\theta^{2n}}, \quad 0 < a < \theta$$

$$\text{var}\, \hat{\theta}_2 = E(\hat{\theta}_2^{\,2}) - (E(\hat{\theta}_2))^2 \qquad E\,\hat{\theta}_2 = \theta$$

$E(\hat{\theta}_2)^2 = ?$

$$E(X_{max}^2) = \int_0^{\theta} a^2 \underbrace{2n \cdot \frac{a^{2n-1}}{\theta^{2n}}}_{f_{max}} da = \frac{2n}{\theta^{2n}} \int_0^{\theta} a^{2n+1} da$$

$$= \frac{2n}{\theta^{2n}} \frac{1}{2n+2} a^{2n+2} \Big|_0^{\theta} = \frac{2n}{2n+2} \cdot \frac{\theta^{2n+2}}{\theta^{2n}}$$

$$= \frac{2n}{2n+2} \theta^2$$

$$E(\hat{\theta}_2)^2 = E\left(\left(\frac{2n+1}{2n} X_{max}\right)^2\right) = E\left(\frac{(2n+1)^2}{(2n)^2} X_{max}^2\right)$$

$$= \frac{(2n+1)^2}{(2n)^2} E X_{max}^2 = \frac{(2n+1)^2}{(2n)^2} \cdot \frac{2n}{2n+2} \cdot \theta^2$$

$$= \frac{(2n+1)^2}{2n \cdot (2n+2)} \theta^2$$

$$\text{var}\, \hat{\theta}_2 = \frac{(2n+1)^2}{2n \cdot (2n+2)} \theta^2 - \theta^2 = \frac{1}{4n(n+1)} \theta^2$$

$$\text{var}\, \hat{\theta}_2 < \text{var}\, \hat{\theta}_1$$

jbradic

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$\text{var } \hat{\theta}_1$ $\quad < ? > \quad$ $\text{var } \hat{\theta}_2$

$\frac{1}{8n}\theta^2 \qquad \frac{1}{4n(n+1)}\theta^2$

$(\ ) \frac{1}{2} \quad \boxed{>} \quad \frac{1}{n+1} (\quad).$

$\frac{1}{2} < \frac{1}{n+1}$? $\quad n+1 < 2.$ $\quad n < 1$?

$\hat{\theta}_2$ is more efficient

Relative efficiency

$$\frac{\text{var}(\hat{\theta}_1)}{\text{var}(\hat{\theta}_2)} = \frac{\cancel{\theta^2}\ \cancel{4}\cancel{n}(n+1)}{\cancel{8n}\ \cancel{\theta^2}} = \frac{n+1}{2}.$$

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$\frac{n+1}{2} > 1$

$n+1 > 2$

$\boxed{n > 1}$ $\qquad \text{var}(\hat{\theta}_1) > \text{var}(\hat{\theta}_2)$ ✓