W8: MONDAY

jbradic

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~~ch 5.3 L&M~~

Cramer-Rao bound (ch 5.5. L&M).

The best estimator, (most efficient estimator) is an unbiased estimator that has variance equal to

$$I^{-1}(\theta) = \left(\mathbb{E}_X\left[\frac{d^2}{d\theta^2} \ln L(\theta) \right] \right).$$

(Fisher-Information matrix). Expected value of second derivative of log-likelihood function.

Ex:

EX: $(X_1, \dots, X_n)$ - SAMPLE FROM DENSITY

$$f_X(x;\theta) = \frac{2x}{\theta^2} \text{ FOR } \boxed{0 \le x \le \theta}.$$

(a) Is MLE ESTIMATOR the most efficient estimator? NO.

$\hat{\theta}_{MLE} = ?$

$$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \frac{2X_i}{\theta^2} \cdot \mathbb{1}(0 \le \underline{\underline{X_i \le \theta}}).$$

$$\boxed{\hat{\theta}_1 = \frac{3}{2n} \sum_{i=1}^{n} X_i, \quad \text{VAR } \hat{\theta}_1 = \frac{1}{8n}\theta^2}.$$

$$\hat{\theta}_{MLE} = \max_{1 \le i \le n} X_i, \qquad \mathbb{E}\,\hat{\theta}_{MLE} = \frac{2n}{2n+1}\theta$$

$\hat{\theta}_{MLE}$ is biased

$\Rightarrow$ hence not the most efficient

(b). Is $\hat{\theta}_1$ most efficient estimator?

$\mathbb{E}\hat{\theta}_1 = \theta$ - $\hat{\theta}_1$ - unbiased estimator.

$$\text{VAR } \hat{\theta}_1 = \frac{1}{8n}\theta^2$$

$$I(\theta) = \mathbb{E}\left[\frac{d^2}{d\theta^2} \ln L(\theta)\right].$$

$$\frac{d}{d\theta} \ln L(\theta) = \frac{d}{d\theta}\left[\ln\left(\prod_{i=1}^{n} \frac{2X_i}{\theta^2}\right)\right] = \frac{d}{d\theta}\left(\sum_{i=1}^{n} \ln \frac{2X_i}{\theta^2}\right).$$

$$= \frac{d}{d\theta}\left(\sum_{i=1}^{n}\left(\ln(2X_i) - 2\ln\theta\right)\right).$$

$$= \frac{d}{d\theta}\left(\sum_{i=1}^{n} \ln(2X_i) - 2n\ln\theta\right).$$

$$\frac{d}{d\theta} \ln L(\theta) = -\frac{2n}{\theta}$$

$$\frac{d^2}{d\theta^2} \ln L(\theta) = \frac{d}{d\theta}\left(\frac{d'}{d\theta} \ln L(\theta)\right)$$

$$= \frac{d}{d\theta}\left(-\frac{2n}{\theta}\right) = +\frac{2n}{\theta^2}$$

Most efficient estimator should have var equal to $\frac{1}{I(\theta)}$

$$I(\theta) = E\left(\frac{d^2}{d\theta^2} \ln L(\theta)\right) = E\left(\frac{2n}{\theta^2}\right) = \frac{2n}{\theta^2}$$

$$\frac{1}{I(\theta)} = \frac{1}{2n} \cdot \theta^2$$

$\hat{\theta}_1 = ?$ $\quad var(\hat{\theta}_1) = \frac{1}{8n}\theta^2$

$var(\hat{\theta}_1) < \frac{1}{I(\theta)}$ - super efficient estimator.

Cramer-Rao bound does not apply because we had $\mathbb{1}$ functions inside L.

thus

If L has 2 derivatives then MLE estimator is the most efficient estimator.

CONFIDENCE INTERVALS (ch 5.3 L&M)

MLE method leads to $\hat{\theta}$ - NO INFO ON INTRINSIC RELIABILITY OF SUCH AN ESTIMATOR

$\Rightarrow$ WE WANT AN INTERVAL

$$[L(\text{Data}), \; U(\text{Data})] = \begin{array}{c}(1-\alpha)\cdot 100\% \text{ CONFIDENCE} \\ \text{INTERVAL (CI)}\end{array}$$

WHICH CONTAINS THE TRUE value OF θ WITH SOME $\underbrace{\text{HIGH probability}}_{1-\alpha}$. α - QUITE SMALL.

$$\mathbb{P}(\text{TRUE } \theta \in [L(\text{Data}), \; U(\text{Data})]) = 1-\alpha.$$

METHOD OF PROOFING!

EX: $(X_1, \dots, X_n)$ - sample that has normal distribution with parameters μ and σ^2

Suppose σ^2 is known to us.

$$\underbrace{\begin{array}{l} X_1 \sim N(\mu, \sigma^2) \\ X_2 \sim N(\mu, \sigma^2) \\ \vdots \\ X_n \sim N(\mu, \sigma^2) \end{array}}_{\text{independent, identically distributed.}}$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

distributions are the same.

~~ide~~ independent, identically distributed.

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, σ^2 - known

μ - unknown

Find 95% CI for μ?

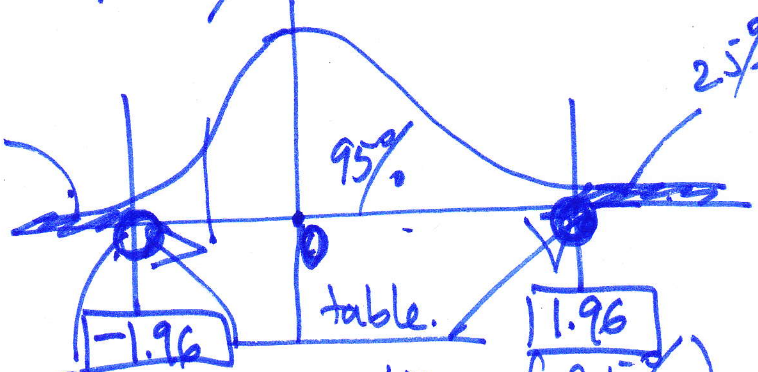
$$\hat{\mu}_{MLE} = \bar{X} \sim N(\mu, \frac{\sigma^2}{n}).$$

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1) \quad (*)$$

95% CI: Looking for L and U such that

$$P(L \le \mu \le U) = 0.95$$

$$P\left(?? \le \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \le ??\right) = 0.95 \quad (**)$$



$z_{0.025}$ (quantile of 2.5%).

$$P\left(-1.96 \le \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \le 1.96\right) = 0.95$$

$$P\left(-1.96 \cdot \frac{\sigma}{\sqrt{n}} - \bar{X} \le -\mu \le 1.96 \frac{\sigma}{\sqrt{n}} - \bar{X}\right) = 0.95$$

$$P\left(\bar{X} - 1.96 \frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + 1.96 \frac{\sigma}{\sqrt{n}}\right) = 0.95$$

95% CI for μ when σ is known

is $\left[\bar{X} - 1.96 \frac{\sigma}{\sqrt{n}},\ \bar{X} + 1.96 \frac{\sigma}{\sqrt{n}} \right]$.