Practice Midterm 2 (183)

Problem 1.

(a) (1) False. It should be smaller

(2) $\hat{\theta}_2$ is more efficient, since $Var(\hat{\theta}_2) = \frac{1}{16} Var(\hat{\theta}_1)$

(3) False. For uniform distribution, say Unif $(0, \theta)$. $X_{(n)}$ is better.

(4) False. From Jensen's inequality.

$E(S) = E\sqrt{S^2} \le \sqrt{E(S^2)} = \sqrt{\sigma^2} = \sigma$, where $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

which is unbiased for $\sigma^2$.

(b) (1) True. Because expectation is a linear operator.

(2) True. Note that $E(X^2)$ is a constant.

(3) True. If $X \sim Bin(n, p)$. X can be written as a sum of

n i.i.d. Ber(p) and then apply C.L.T.

(4). False. It's true provided the distribution has finite variance.

A counterexample: $X_i \overset{iid}{\sim}$ Cauchy distribution.

(c). (1) Suppose $X \sim$ exponential. We have $P(X > s+t \mid X > t) = P(X > s)$,

for all s, t strictly positive.

(2) True. The property of normal distribution

(3) False. Consider $X = Y \sim Pois(\lambda)$. $X + Y$ can only take even integers. But a poisson r.v. takes any nonnegative integer with positive probability. So $X+Y$ can't be poisson.

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Problem 2
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(a) We assume that subjects of the study will disobey authority each with probability 0.35 and independently. Under this assumption, we have a binomial distribution with parameters $n = 20$ and $p = .35$. Call this random variable $X$. Then the question asks for the probability that $X$ is greater than 3, which is equal to $1 - P(X \leq 3)$. By additivity, this is equal to $1 - (P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3))$. Since $X$ is a binomial distribution, this is given by

$$1 - \sum_{k=0}^{3} \binom{20}{k} (0.35^k)(0.65)^{20-k}.$$

(b) Similar to above, we can give the probability as

$$\sum_{k=30}^{40} \binom{400}{k} (0.35^k)(0.65)^{400-k}.$$

However, when $n$ is large, the binomial distribution approximates the normal distribution well, so we can say that $X$ is approximated by $N(np, np(1-p)) = N(140, 91)$. Then

$$P(30 < X < 40) \approx P(30 < N < 40) = P\left(\frac{30-140}{\sqrt{91}} < Z < \frac{40-140}{\sqrt{91}}\right) \approx P(-11.1 < Z < -10$$

This probability is very close to 0.

(c) We are now interested in the probability that it takes more than 5 attempts for a success when each trial has probability 0.35 of success independently. Thus if $X$ is a geometric distribution with parameter 0.35, we are interested in $P(X > 5)$.

$$P(X > 5) = 1 - P(X \leq 5) = 1 - \sum_{k=1}^{5} (0.65)^{k-1}(0.35).$$

(d) This is just asking the expected value of a geometric distribution with parameter (0.35), which is given by $\frac{1}{p} = \frac{1}{0.35}$.

Problem 3. $X_1, \cdots, X_n \overset{iid}{\sim} f_X(x;\beta) = \dfrac{x^4}{24\beta^5} e^{-x/\beta}, \quad x > 0.$

(a) The likelihood function is

$$L(\beta) = \prod_{i=1}^{n} f_{X_i}(x_i;\beta) = \prod_{i=1}^{n} \frac{x_i^4}{24\beta^5} e^{-x_i/\beta} = \underbrace{\left(\prod_{i=1}^{n} \frac{x_i^4}{24}\right)}_{\text{a const.}} \cdot \frac{1}{\beta^{5n}} e^{-\frac{\sum_{i=1}^{n} x_i}{\beta}}$$

$$\log L(\beta) = \text{const} - 5n \log \beta - \frac{\sum_{i=1}^{n} x_i}{\beta}$$

Setting

$$\frac{d}{d\beta} \log L(\beta) = -\frac{5n}{\beta} + \frac{\sum_{i=1}^{n} x_i}{\beta^2} = 0 \quad \text{gives} \quad \beta_e = \frac{\sum_{i=1}^{n} x_i}{5n}.$$

Hence MLE is $\hat{\beta} = \dfrac{\sum_{i=1}^{n} X_i}{5n}$, (which is a r.v.)

(b) $$E(\hat{\beta}) = E\left(\frac{\sum_{i=1}^{n} X_i}{5n}\right) = \frac{1}{5n}\sum_{i=1}^{n} E(X_1) = \frac{1}{5n}\sum_{i=1}^{n} 5\beta = \frac{5n\beta}{5n} = \beta.$$

(note $E(X_1) = \cdots = E(X_n)$, since they have the same distribution)

(c) $$Var(\hat{\beta}) = Var\left(\frac{\sum_{i=1}^{n} X_i}{5n}\right) = \frac{1}{25n^2}\sum_{i=1}^{n} Var(X_i)$$, since $X$'s are independent and variance of sum is a sum of variance.

$$Var(\hat{\beta}) = \frac{1}{25n^2} \cdot n\, Var(X_1) = \frac{5\beta^2}{25n} = \frac{\beta^2}{5n}.$$

(note $Var(X_1) = \cdots = Var(X_n)$, since they have the same distribution).

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Problem 4
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(a) I will assume that the problem states, "given $Z = 8$" instead of 80 as $Z$ can take a maximum value of 30. Then

$$P(X = 5|Z = 8) = \frac{P(X = 5 \text{ and } Y = 3)}{P(X + Y = 8)}$$

Since $X$ and $Y$ are independent, the numerator is $P(X = 5)P(Y = 3) = \binom{10}{5}p^5(1-p)^5 \cdot \binom{20}{3}p^3(1-p)^{17}$. Now $P(X + Y = 8)$ is given by the sum of the probabilities that $X = 0$ and $Y = 8$, $X = 1$ and $Y = 7$,... So

$$P(X = 5|Z = 8) = \frac{\binom{10}{5}p^5(1-p)^5 \cdot \binom{20}{3}p^3(1-p)^{17}}{\sum_{k=0}^{8}\binom{10}{k}p^k(1-p)^{10-k}\binom{20}{8-k}p^{8-k}(1-p)^{20-(8-k)}}.$$

(b) For this problem, we use heavily that there are only a few ways to factor 10. That is, 10 can be factored as $10 \cdot 1$ or $5 \cdot 2$. So for $XZ = 10$, we have only four cases

(a) $X = 1$ and $Z = 10$.
(b) $X = 10$ and $Z = 1$.
(c) $X = 5$ and $Z = 2$.
(d) $X = 2$ and $Z = 5$.

Now, since $Z \geq X$, cases 2 and 3 cannot ever occur. Thus the probability that $XZ = 10$ is the sum of the probabilities that cases 1 and 4 occur. $P(X = 1 \text{ and } Z = 10) = P(X = 1 \text{ and } Y = 9) = P(X = 1)P(Y = 9) = \binom{10}{1}p_1^1(1-p_1)^9 \cdot \binom{20}{9}p_2^9(1-p_2)^{11}$, and $P(X = 2 \text{ and } Z = 5) = P(X = 2 \text{ and } Y = 3) = P(X = 2)P(Y = 3) = \binom{10}{2}p_1^2(1-p_1)^8 \cdot \binom{20}{3}p_2^3(1-p_1)^{17}$. So

$$P(XZ = 10) = \binom{10}{1}p_1^1(1-p_1)^9 \cdot \binom{20}{9}p_2^9(1-p_2)^{11} + \binom{10}{2}p_1^2(1-p_1)^8 \cdot \binom{20}{3}p_2^3(1-p_1)^{17}$$