W9: ~~Monday~~ Wednesday

Hypotheses testing (ch 6).

$X_1, \dots, X_n \sim$ iid $f_\theta \in \{ f_\theta : \theta \in \Theta \}$.

$\hat{\theta}$ ✓.

CI for θ ✓

[For some θ_0]

Is data reasonably compatible with Θ_0?

(not "do we believe $\theta = \theta_0$ exactly")

or is the evidence against $\theta = \theta_0$ strong?

$H_0: \theta = \theta_0$ (null)

$H_1: \theta \neq \theta_0$ (2-sided alternative)

To fix ideas suppose

$X_1, \dots, X_n \sim N(\mu, \sigma_0^2)$

μ unknown, σ_0^2 known.

test:

$H_0: \mu = \mu_0$, μ_0 - known

vs. $H_1: \mu > \mu_0$ (one-sided alternative)

Decision $\begin{cases} \nearrow H_0 \\ \searrow H_1 \end{cases}$ but language is delicate

And decision may be wrong either way

We'll either "accept H_0" in favor of H_1

or

"reject H_0"

based on ~~some~~ a test-statistic

Test statistic is usually connected to MLE estimator of the parameter we are testing i.e. of μ.

Suppose the decision rule was

reject H_0 iff $\bar{X} > \mu_0$

Primary fear:

False rejection - type I error

$\mathbb{P}(\bar{X} > \mu_0 \mid$ when μ is really equal to $\mu_0)$

$$= \mathbb{P}\left(\frac{\bar{X} - \mu_0}{\sigma_0^2/\sqrt{n}} > 0 \right) = \underbrace{0.5}_{\text{unacceptably high}}$$

Hence boundary needs margin on the right

Suppose we are willing to tolerate some false rejection.

level of a test

$$\mathbb{P}(\text{false rejection}) = \boxed{\alpha}$$

α - some small number (often 5%, 1%)

$$\mathbb{P}(\text{reject} \mid \mu = \mu_0) = \mathbb{P}(\bar{X} > \mu_0 + ?) = \alpha.$$

$$\Rightarrow \mathbb{P}\left(\frac{\bar{X} - \mu_0}{\sigma_0/\sqrt{n}} > \underbrace{\frac{?}{\sigma_0/\sqrt{n}}}_{z_{1-\alpha}}\right) = \alpha.$$

$\Rightarrow$ Hence rejection rule is:

Reject iff

$$\bar{X} > \underbrace{\mu_0 + \frac{\sigma_0}{\sqrt{n}} z_{1-\alpha}}_{\text{gets closer to } \mu_0 \text{ with } n \to \infty}.$$

When $\alpha \to 0$: harder to reject

$n \to \infty$: easier to reject

$\sigma_0 \to \infty$ harder to reject

Of course, we want to reject when we should. Does this procedure ensures of that?

→ Type II error:

$$\mathbb{P}[\text{type II error}] =$$

$$= \mathbb{P}\left(\bar{X} < \mu_0 + \frac{\sigma_0}{\sqrt{n}} z_{1-\alpha} \,\middle|\, \mu = \mu_1 \right)$$

point inside H_1

$$= \mathbb{P}\left(\frac{\bar{X} - \mu_0}{\sigma_0/\sqrt{n}} < \frac{\mu_0 - \mu_1 + \frac{\sigma_0}{\sqrt{n}} \cdot z_{1-\alpha}}{\sigma_0/\sqrt{n}} \right)$$

$$= \Phi\left(\frac{\mu_0 - \mu_1}{\sigma_0/\sqrt{n}} + z_{1-\alpha} \right)$$

Ideally we want Type II error → 0