W10: MONDAY.

READING: 6.2, 6.3, 6.4
7.4, 7.5, 7.6

EX: $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma_0^2)$

μ unknown, σ_0^2 known.

$H_0: \mu = \mu_0$

$H_1: \mu > \mu_0$

$\{\bar{X} > \mu_0\}$ - not good enough.

Type I error = α ← Use it to derive critical region

Type II error = β

$$\left\{ \bar{X} > \mu_0 + \frac{\sigma_0^2}{\sqrt{n}} \cdot z_{1-\alpha} \right\}.$$

Ideally $\alpha = 0$, & $\beta = 0$. Unfortunately this cannot happen.

$\left\{ \begin{array}{l} \text{When } \alpha \text{ is increased } \uparrow, \ \beta \text{ decreases.} \\ \text{and } \alpha \text{ is decreased } \downarrow, \ \beta \uparrow \end{array} \right\}.$

— Fix α to be say 1% (5%), minimize β as much as possible.

Comparing II tests:

If the same α — then we prefer the test with smaller β

If α's are different — then we prefer the test with smaller α.

$H_0: \mu = \mu_0$

$H_1: \mu > \mu_0$

$$P(\text{type II error}) = \beta$$

$$= P(\underbrace{\text{accept } H_0}_{\text{complement of critical region}} \mid \underbrace{H_0 \text{ not true}}_{\text{true } \mu = \text{ some } \mu_1 \text{ for } \mu_1 > \mu_0})$$

$$\bar{X} \le \mu_0 + \frac{\sigma_0}{\sqrt{n}} z_{1-\alpha}.$$

β - always depends on μ_1 (point in the alternative).

$$\beta = P\left(\bar{X} \le \mu_0 + \frac{\sigma_0}{\sqrt{n}} z_{1-\alpha} \mid \mu = \mu_1\right)$$

$$\bar{X}\Big|_{H_0} \sim N\left(\mu_0, \frac{\sigma_0^2}{n}\right)$$

$$\boxed{\bar{X}\Big|_{H_1} \sim N\left(\mu_1, \frac{\sigma_0^2}{n}\right)}$$

$$\beta = \mathbb{P}\left(\frac{\bar{X} - \mu_1}{\sigma_0/\sqrt{n}} \le \frac{\mu_0 + \frac{\sigma_0}{\sqrt{n}} z_{1-\alpha} - \mu_1}{\frac{\sigma_0}{\sqrt{n}}} \right).$$

$$= \mathbb{P}\left(Z \le \frac{\mu_0 - \mu_1}{\frac{\sigma_0}{\sqrt{n}}} + z_{1-\alpha} \right).$$

If
$\mu_0 = 0.5$
$\mu_1 = 1$
$\alpha = 5\%$
$\sigma_0 = 1$
$n = 100$

$$\Rightarrow \beta = \mathbb{P}\left(Z \le \frac{\overset{(0.5-1)}{\overset{\parallel}{-0.5}}}{1/10} + \underset{\underset{1.64}{\sim}}{z_{0.95}} \right)$$

$$\beta = \mathbb{P}(Z \le -5 + 1.64)$$

Power of a test: $1 - \beta$ for any $\underline{\mu_1}$.

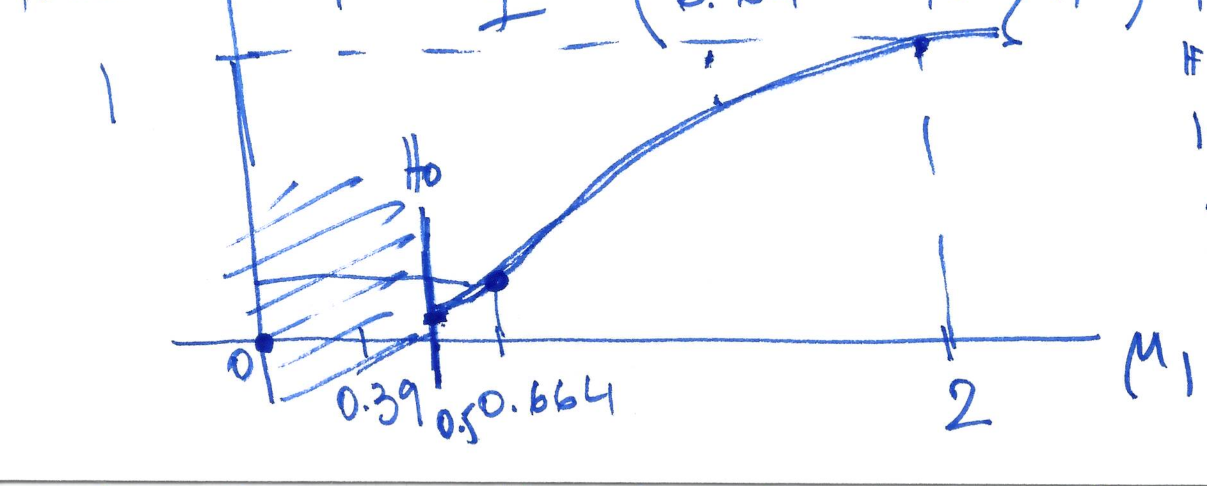
$$1 - \Phi\left(\frac{0.5 - \mu_1}{1/\cancel{1}10} + z_{0.95} \right)$$

$$= 1 - \Phi\left(+5 - 10 \cdot \mu_1 + 1.64 \right) \;??$$

$$= 1 - \Phi\left(6.64 - 10 \cdot \mu_1 \right)$$

0.39?

If $\mu_1 = 0.664$
$1 - \Phi(0)$
$= 0.5$



$H_0: \mu = \mu_0$

$H_1: \mu < \mu_0$

Rejection region $\left\{ \bar{X} < \mu_0 + \frac{\overset{\text{known}}{\sigma_0}}{\sqrt{n}} z_{1-\alpha} \right\}$.

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$

Rejection region $\left\{ |\bar{X}| > \mu_0 + \frac{\overset{\text{known}}{\sigma_0}}{\sqrt{n}} \cdot z_{1-\frac{\alpha}{2}} \right\}$

$$\bar{X} \notin \left[\mu_0 \pm \frac{\sigma_0}{\sqrt{n}} z_{1-\frac{\alpha}{2}} \right]$$

- What to do if σ_0 is unknown.
  - Replace σ_0 in previous calculations with $\hat{\sigma} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2}$
  - Replace z's with t's with $n-1$ d.f.

Ex: $\bar{x} \notin \left[\mu_0 \pm \frac{\hat{\sigma}}{\sqrt{n}} \cdot t_{1-\frac{\alpha}{2}, n-1} \right]$.

- What if $X_1, \dots, X_n \sim$ Binomial (n, p)

known unknown.

$$\hat{p}_{MLE} = \frac{X}{n}, \quad X \sim B(n,p)$$

$$\boxed{\frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \sim Z \text{ distribution asymptotically.}}$$

$$\boxed{\frac{\hat{p} - p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \approx Z\text{-distribution}}$$

$H_0: p = p_0\ 0.5$

$H_1: p > p_0$

Rejection Region? too complicated

- Write type I error.

$\mathbb{P}($ Rejecting H_0 | when H_0 is true $) = \alpha$.

$$\mathbb{P}\left(\hat{p}_{MLE} > C^{??} \mid p = p_0\right) = \alpha.$$

$$\mathbb{P}\left(\underbrace{\frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}}_{Z} > \underbrace{\frac{C - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}}_{Z_{1-\alpha}}\right) = \alpha.$$

$$\Rightarrow C = p_0 + Z_\alpha \sqrt{\frac{p_0(1-p_0)}{n}}$$