`Problem 1`

(a) To make a decision rule, usually we first consider Type I error, which is:

$$\begin{aligned} P(\text{Reject } H_0|H_0 \text{ is true}) &= P(\hat{\mu} < c|\mu \geq 10) \\ &= P\Big(\frac{\hat{\mu}-\mu}{\sigma_0/\sqrt{n}} < \frac{c-\mu}{\sigma_0/\sqrt{n}}\Big|\mu \geq 10\Big) \\ &\leq P\Big(\frac{\hat{\mu}-\mu}{\sigma_0/\sqrt{n}} < \frac{c-\mu}{\sigma_0/\sqrt{n}}\Big|\mu = 10\Big) \end{aligned}$$

Our purpose is to find $c$ to make Type I error become $\alpha$ (which is 1% here). From the conclusion above, we find type I error is always smaller than error when $\mu = 10$, if we find $c$ to make the latter error be $\alpha$, then type I error will always be smaller than $\alpha$.

$$\begin{aligned} 0.01 &= P\Big(\frac{\hat{\mu}-\mu}{\sigma_0/\sqrt{n}} < \frac{c-\mu}{\sigma_0/\sqrt{n}}\Big|\mu = 10\Big) \\ &= P\Big(\frac{\hat{\mu}-10}{\sigma_0/\sqrt{100}} < \frac{c-10}{\sigma_0/\sqrt{100}}\Big) \\ &= P\Big(z < \frac{c-10}{\sigma_0/10}\Big). \end{aligned}$$

This implies:

$$\frac{c-10}{\sigma_0/10} = z_{0.01} \Rightarrow c = 10 + z_{0.01}\frac{\sigma_0}{10}$$

And the hypothesis test is:

$$\hat{\mu} = \bar{X} < 10 + z_{0.01}\frac{\sigma_0}{10}$$

(b) If $sigma_0$ is unknown, the common method is plug in the estimator of $\sigma_0$, which is sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Use the same argument in part(a), the test is same as test for hypothesis:

$$H_0 : \mu = 10; H_1 : \mu < 10$$

Then we can directly use theorem 7.4.2, the test is $t = \frac{\bar{X}-10}{S/\sqrt{n}}$, and

$$t \leq -t_{0.99,99}$$

Where $-t_{0.99,99}$ is the quantile of t-distribution.

(c) Yes. One reason is when $n$ is large, the sample variance $S^2$ converges to $\sigma_0^2$. Another one is, t-distribution converges to normal distribution as the freedom (n-1) (where n is just sample size) going to infinity.

(d) Type II error is when $H_1$ is true, we fail to reject $H_0$.

$$\begin{aligned} P(\text{Fail to reject } H_0|\mu=5) &= P(\hat{\mu} \geq c|\mu=5) \\ &= P\Big(\frac{\hat{\mu}-\mu}{\sigma_0/\sqrt{n}} \geq \frac{c-\mu}{\sigma_0/\sqrt{n}}\Big|\mu=5\Big) \\ &= P\Big(z \geq \frac{\sqrt{n}}{\sigma_0}\Big(10+z_{0.01}\frac{\sigma_0}{\sqrt{n}}-5\Big)\Big) \\ &= P(z \geq z_{0.01}+50/\sigma_0) \end{aligned}$$

(e) Power is $1-\beta$, where $\beta$ is just type II error. If the true $\mu=\mu_1$, where $\mu_1$ is a constant smaller than 10. Then:

$$\begin{aligned} \beta &= P(\text{Fail to reject } H_0|\mu=\mu_1) \\ &= P\Big(z \geq z_{0.01}+\frac{10-\mu_1}{\sigma_0}\Big) \end{aligned}$$

So the power is:

$$1-P\Big(z \geq z_{0.01}+\frac{10-\mu_1}{\sigma_0}\Big)$$

`Problem 2`

To find whether this is a good test or not, we need to consider both Type I error and Type II error.

$$\begin{aligned} P(\text{Reject } H_0|H_0 \text{ is true}) &= P\Big(\hat{p} > 0.15+\sqrt{\frac{0.9\cdot0.1}{10}}1.5\Big|p \leq 0.1\Big) \\ &= P\Big(\frac{\hat{p}-p}{\sqrt{p(1-p)/n}} > \frac{0.15+\sqrt{\frac{0.9\cdot0.1}{10}}1.5-p}{\sqrt{p(1-p)/n}}\Big|p \leq 0.1\Big) \\ &\leq P\Big(\frac{\hat{p}-p}{\sqrt{p(1-p)/n}} > \frac{0.15+\sqrt{\frac{0.9\cdot0.1}{10}}1.5-p}{\sqrt{p(1-p)/n}}\Big|p = 0.1\Big) \\ &\approx P\Big(z > \frac{0.15+\sqrt{\frac{0.9\cdot0.1}{10}}1.5-0.1}{\sqrt{0.1\cdot0.9/100}}\Big) \\ &= P(z > 6.41) = 7.27\times10^{-11} \end{aligned}$$

Although Type I error is so small, we can see type II error, assume the true $p = p_1$, where $p_1 > 0.1$

$$\begin{aligned}
P(\text{Fail to reject } H_0|p = p_1) &= P\Big(\hat{p} \leq 0.15 + \sqrt{\frac{0.9\cdot 0.1}{10}}1.5\Big|p = p_1\Big) \\
&= P\Big(\frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \leq \frac{0.15 + \sqrt{\frac{0.9\cdot 0.1}{10}}1.5 - p}{\sqrt{p(1-p)/n}}\Big|p = p_1\Big) \\
&\approx P\Big(z \leq \frac{0.15 + \sqrt{\frac{0.9\cdot 0.1}{10}}1.5 - p_1}{\sqrt{p_1(1-p_1)/100}}\Big)
\end{aligned}$$

If $p_1 = 0.15$, then:

$$\begin{aligned}
&= P\Big(z \leq \frac{0.15 + \sqrt{\frac{0.9\cdot 0.1}{10}}1.5 - 0.15}{\sqrt{0.15\cdot 0.85/100}}\Big) \\
&= \Phi(3.99) \approx 1
\end{aligned}$$

We can see Type II error will happen with probability 1, this is terrible. So this test is not good at all.

`Problem 4, LM 6.4.8`

We compute the probabiity of a Type II error given that $\mu = 12$ and $\sigma = 4$. First, for a Type II error to occur, we must have not had enough evidence to reject the null hypothesis, which means

$$-1.96 < \frac{\overline{X} - 10}{4/\sqrt{45}} < 1.96$$

which means that

$$8.83 \approx 10 - 1.96\frac{4}{\sqrt{45}} < \overline{X} < 10 + 1.96\frac{4}{\sqrt{45}} \approx 11.17$$

Then we have

$$P(\text{Type II error}|\mu = 12) = P(\overline{X} \in [8.83, 11.17]|\mu = 12).$$

To calculate this probability, we normalize, and have

$$\begin{aligned}
&P(8.83 \leq \overline{X} \leq 11.17|\mu = 12) = P\left(\frac{8.83 - 12}{4/\sqrt{45}} \leq Z \leq \frac{11.17 - 12}{4/\sqrt{45}}\right) \\
&\approx P(-5.31 < Z < -1.39) \approx P(Z < -1.39) = .0823.
\end{aligned}$$

So there is about an 8 percent chance of Type II error, and 45 is a large enough sample.

`Problem 5, LM 6.4.18` Note that we cannot use the normal distribution in this problem:

(a) Let $X$ be the random variable that gives the value of our datum from the Poisson distribution. From Example 5.6.1, $X$ is a sufficient statistic for $\lambda$. The a Type I error occurs when $H_0$ is true (that is, $\lambda = 6$, but we reject the null hypothesis (that is, $X \leq 2$). If $\lambda = 6$, the probability that this happens is

$$P(X \leq 2) = \sum_{k=0}^{2} e^{-6}\frac{6^k}{k!} \approx .06.$$

(b) If $\lambda = 4$, the probability of a Type II error is the probability that we do not reject the null hypothesis, which is the probability that $X > 2$. So

$$P(\text{Type II error}|\lambda = 4) = \sum_{k=3}^{\infty} e^{-4}\frac{4^k}{k!} = 1 - \sum_{k=0}^{2} e^{-4}\frac{4^k}{k!} \approx .76$$

`Problem 6, LM 7.4.20`

I typed all the data into R, stored as a vector x. mean(x) = 0.637; var(x) = 0.02, so s = 0.1414. The t score is thus $\frac{.637-.618}{\frac{.1414}{\sqrt{34}}} = 0.8$. Now look up a table of critical values for the t distribution online. If you have $\alpha = 0.01$ and d.f. $= 34 - 1 = 33$, then the critical value is 2.73. Our t-score was less than that. Therefore, we do not reject $H_0$.

`Problem 7, LM 7.5.14`

Let's show that $\bar{Y}$ is the MLE for $\theta$. $L(\theta) = \prod \frac{1}{\theta}e^{-\frac{y_i}{\theta}} = \theta^{-n}e^{-\frac{1}{\theta}\sum y_i}$.
$\ln(L(\theta)) = -n\ln(\theta) - \frac{1}{\theta}\sum y_i$. Take derivative to get $-\frac{n}{\theta} + \frac{1}{\theta^2}\sum y_i$. Setting that equal to 0 and solving, we get $n\theta^2 = \theta \sum y_i$, so $\theta = \frac{1}{n}\sum y_i$.

According to part a, $\frac{2n\bar{Y}}{\theta}$ is chi square 2n df distributed. Thus, by definition, $P(\chi^2_{\alpha/2,2n} \leq \frac{2n\bar{Y}}{\theta} \leq \chi^2_{1-\alpha/2,2n}) = 1 - \alpha$. If we take the reciprocal and multiply through by $2n\bar{Y}$, we get our interval: $\theta \in \left[\dfrac{2n\bar{Y}}{\chi^2_{1-\alpha/2,2n}}, \dfrac{2n\bar{Y}}{\chi^2_{\alpha/2,2n}}\right]$ with probability $1 - \alpha$.

`Problem 8, LM 7.5.16`

$(n-1)s^2 = (n-1)\frac{1}{n-1}\sum(y_i-\bar{y})^2 = \sum(y_i^2-2y_i\bar{y}+\bar{y}^2) = (\sum y_i^2)-2(n\bar{y})\bar{y}+n\bar{y}^2 = (\sum y_i^2)-n\bar{y}^2$, which is easily computed from the numbers provided by the text. I get 12.32. So that would be $\chi^2$, since $\sigma_0^2 = 1$. We reject if $\chi^2 \geq \chi^2_{1-.05,30-1}$. Looking that up in a table, we see it's 42.56, so we don't reject.