Differential Equations
with
Mathcad Prime

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These are my notes on ordinary differential equations, worked out in Mathcad Prime 2.0. They only contain the basics - no engineering examples and no exercises. Feel free to use and improve them.

Hans Wesselingh, Groningen, April 2012

Contents

References

Hans Wesselingh and Hans de Waard, *Calculate and Communicate with Mathcad Prime,* VSSD Publishers, Delft 2012 (www.vssd.nl/hlf)
Erwin Kreyszig, *Advanced Engineering Mathematics*, John Wiley & Sons (many editions)

1 Introduction (Terminology)

In this first lesson, we look at names and terms in differential equations.

Some Terms

Here is a simple differential equation: $\frac{d}{dx}y(x) = k \cdot x$

It contains two variables *x* and *y* and a constant *k*. We will regard *y* as a function of *x* - it is the *dependent* variable. The other variable *x* is the *independent* one.

The left side of the equation only contains a first derivative: this gives a *first order* equation. The derivative can be regarded as the quotient of two differentials *dx* and *dy*. A derivative is often written using prime notation *y'*(*x*) and in this subject also using the symbol *p* (parameter):

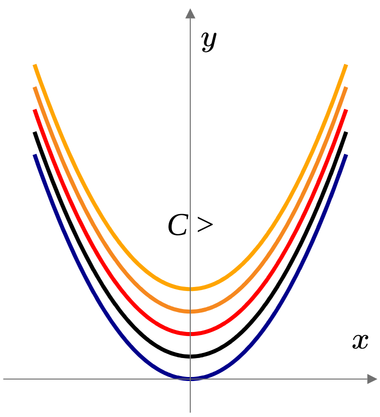
$$\frac{d}{dx}y(x) = \frac{dy}{dx} = y'(x) = p$$

We will use all these notations - choosing the simplest for a given problem.

The Solution

The *solution* to our equation is most easily obtained using differential notation:

$$dy = k \cdot x \cdot dx$$



We can integrate both sides to get:

$$y = k \cdot \frac{x^2}{2} + C$$

This solution is a function that does not contain any derivative. It does contain an integration constant *C*. This can have any value - the equation has an infinite number of solutions. The set is known as the *general solution*. If you give *C* some value, you have a *particular* solution.

The plot here shows some particular solutions. In this simple example a single solution runs through each point of the *x*-*y* plane.

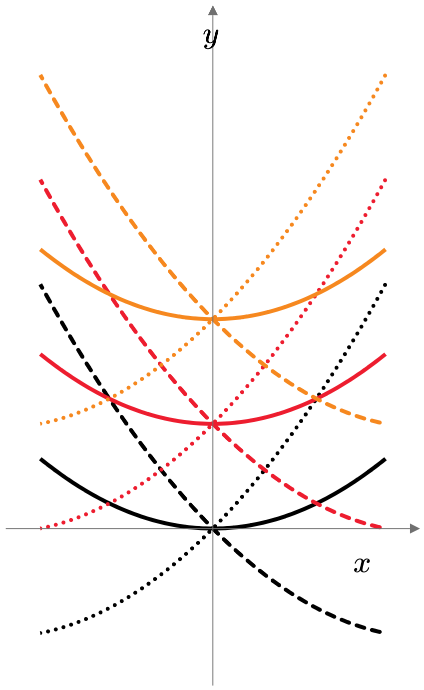
The Order

The following equation contains a second derivative. It is a *second order* equation.

$$\frac{d^2}{dx^2}y(x) = k \cdot x$$

You can obtain the solution by integrating twice:

$$\frac{d}{dx}y(x) = k \cdot \frac{x^2}{2} + C_1 \qquad y(x) = k \cdot \frac{x^3}{6} + C_1 \cdot x + C_2$$



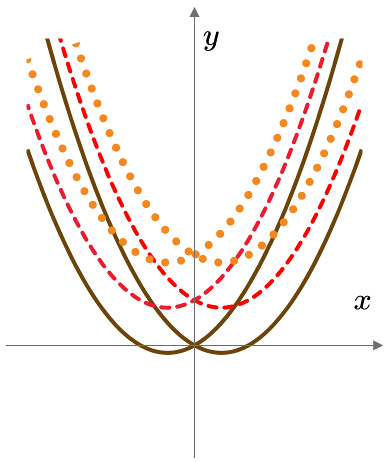
This equation has *two* integration constants. Here there are two solutions running through each point in the *x*,*y* plane.

The crossing black lines are for three values of the first constant; the three colours are for three values of the second constant.

The Degree

Our third equation contains the square of a *first* derivative. It is a *first order* equation, but of *degree two*.

$$\left(\frac{d}{dx}y(x)\right)^2 = x^2 - 1$$



You can regard this as the product of two differential equations:

$$\frac{d}{dx}y(x) = x - 1 \qquad \frac{d}{dx}y(x) = x + 1$$

Each solution is the product of the two solutions:

$$y_1(x) - \frac{x^2}{2} + x - C = 0$$
$$y_2(x) - \frac{x^2}{2} - x - C = 0$$

For each value of the integration constant we get a 'double line' on which the product is zero.

Categories

The first two examples are *linear* equations - the third is non-linear. A non-linear equation contains some product of two expressions of the *dependent* variable *y*(*x*). See whether you understand the difference:

(1)
$$\frac{d}{dx}y(x) = k \cdot x$$

(2)
$$\frac{d^2}{dx^2}y(x) = k \cdot x$$

(3)
$$\left(\frac{d}{dx}y(x)\right)^2 = x^2 - 1$$

Methods

There are two main groups of methods for solving differential equations:
(a) the *symbolic* methods, leading to equations
(b) the *numerical* methods, leading to numbers.

In the examples above, we have used symbolic methods. When these work, they are fine. They can give an overview that is difficult to obtain numerically. However, most differential equations to not have a symbolic solution. If a solution exists, finding it often requires puzzling and patience.

Numerical methods are more flexible. However, they also have their problems. To become a good differential-equation-solver you will need to master both methods. We begin with the symbolic methods.

2 First Order, First Degree Equations

The simplest differential equations are those of the first order and first degree. Over the centuries many tricks have evolved to solve these.

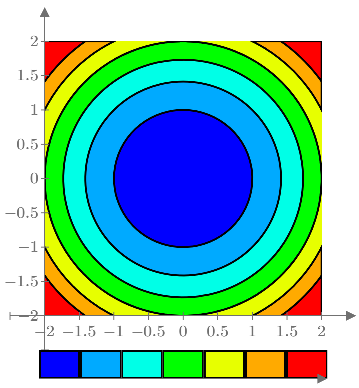
Separable Equation

If each of the terms in a differential equation only contains a single variable, then the terms can be integrated directly. As an example we consider:

$$\frac{dy}{dx} = -\frac{y}{x} \quad \text{or} \quad x \cdot dx + y \cdot dy = 0$$

$$\int x\,dx \rightarrow \frac{x^2}{2} \qquad \int y\,dy \rightarrow \frac{y^2}{2}$$

Including the integration constant: $\frac{x^2}{2} + \frac{y^2}{2} = C$



You can regard any particular solution as the intersection of the 3D function:

$$f(x, y) = x^2 + y^2$$

with a plane of height *C*. This is one of the lines of constant height that you see in a contour plot. In this example these are circles around the origin.

Integration Constant

It is important to include the integration constant immediately on integration. This example shows why.

$$\frac{dy}{dx} = 1 + y^2 \qquad \frac{dy}{1+y^2} = dx \qquad \operatorname{atan}(y) = x + C \qquad y(x, C) := \tan(x + C)$$

We check that this is indeed a solution:

$$\frac{d}{dx} y(x, C) \rightarrow \tan(C + x)^2 + 1 \qquad \frac{dy}{dx} = 1 + y^2$$

If we include the constant after obtaining *y*, we get the incorrect:

$$Y(x, C) := \tan(x) + C$$

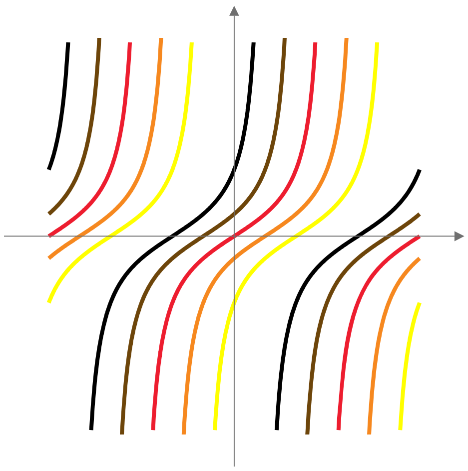
Checking gives

$$\frac{d}{dx} Y(x, C) \rightarrow \tan(x)^2 + 1 \qquad \frac{dY}{dx} = 1 + (Y - C)^2$$

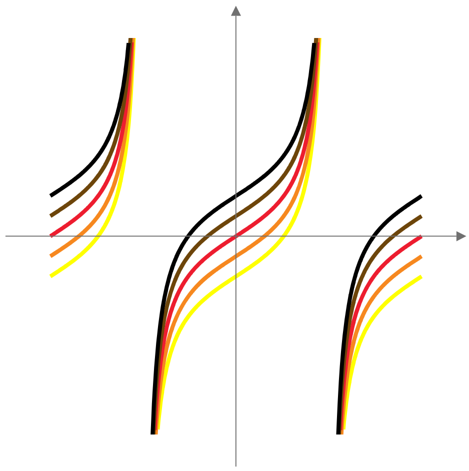
This is not the original equation.

A set of solutions is shown below.

y, correct



Y, incorrect



Ratio Equation

You can often bring a first order equation into a separable form using a substitution. One group of equations where this works are the 'ratio' equations. These can be written in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

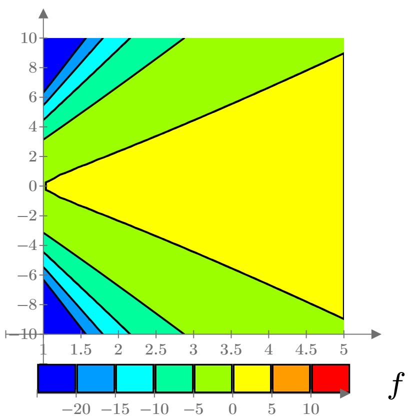
(The 'circles' example is a ratio equation, be it a very simple one.) If you replace *x* and *y* in an equation by *kx* and *ky*, and the *k*'s cancel in the result, then you have a ratio equation. These are brought into a separable form using the substitution:

$$\frac{y}{x} = z \qquad \text{or} \qquad y = z \cdot x \qquad dy = x \cdot dz + z \cdot dx$$

Take the ratio equation: $\frac{dy}{dx} = \frac{x^2 + y^2}{x \cdot y}$

Using differential notation: $x \cdot y \cdot dy - x^2 \cdot dx - y^2 \cdot dx = 0$

The substitution gives $\left(-x^2\right) \cdot dx + x^3 \cdot z \cdot dz$ $\quad \left(\frac{dx}{x}\right) = z \cdot dz$



$$\ln(x) = \frac{z^2}{2} + C$$

The general solution becomes

$$\ln(x) = \frac{(y \div x)^2}{2} + C$$

It is not easy to simplify this, but the contour plot shows particular solutions. (To get the plot, I had to reduce the *x*-range.)

Linear Equation

The linear first order differential equation is: $\frac{dy}{dx}+P(x)\cdot y+Q(x)=0$

As an example we consider $\frac{dy}{dx}-x\cdot y+x^2=0$ or $dy=\left(x\cdot y-x^2\right)\cdot dx$

This can be solved using the substitution: $y=u(x)\cdot v(x)$ $dy=v\cdot du+u\cdot dv$

So $v\cdot du+u\cdot dv=\left(u\cdot v-x^2\right)\cdot dx$ or $v\cdot(du-u\cdot dx)+u\cdot dv=-x^2\cdot dx$

We can choose one of the two functions *u*(*x*) and *v*(*x*). Here we choose *u* such that the first term in the last equation becomes zero:

$$du-u\cdot dx=0 \qquad \frac{du}{u}=dx \qquad \ln(u)=x \qquad u=e^x$$

(There is no need to include an integration constant here.) The remaining equation becomes

$$u\cdot dv=-x^2\cdot dx \qquad dv=-x^2\cdot e^{-x}\cdot dx \qquad -\int x^2\cdot e^{-x}\,dx\rightarrow e^{-x}\cdot\left(x^2+2\cdot x+2\right)$$

So $v=e^{-x}\cdot\left(x^2+2\cdot x+2\right)+C$

and the general solution is $u\cdot v=x^2+2\cdot x+2+C$

We will see other methods for linear equations when we consider second order equations.

clear (x,y)

'Exact' Equation

This method is easiest to understand by starting with the *solution* to a differential equation and working backwards. The solution that we choose is:

$$f(x,y,C)=0 \qquad f(x,y,C):=x+\cos(x\cdot y)+C$$

The differential of this is: $df=\frac{df}{dx}\cdot dx+\frac{df}{dy}\cdot dy$

(With partial differential quotients. Mathcad does not have symbols for these.)

With

$$\frac{d}{dx}f(x,y,C)\rightarrow 1-y\cdot\sin(x\cdot y)$$

$$\frac{d}{dy}f(x,y,C)\rightarrow -(x\cdot\sin(x\cdot y))$$

we obtain the differential equation

$$(1-y\cdot\sin(x\cdot y))\cdot dx-(x\cdot\sin(x\cdot y))\cdot dy=0$$

If we *start* with this equation, how do we obtain the solution? We first check that it is indeed a differential. This is so when:

y-derivative of first factor = *x*-derivative of second factor

$$\frac{d}{dy}(1-y\cdot\sin(x\cdot y)) \rightarrow -\sin(x\cdot y)-x\cdot y\cdot\cos(x\cdot y)$$

$$\frac{d}{dx}-(x\cdot\sin(x\cdot y)) \rightarrow -\sin(x\cdot y)-x\cdot y\cdot\cos(x\cdot y)$$

The two are indeed equal, so the equation is a differential. We then know:

$$\frac{df}{dx}=1-y\cdot\sin(x\cdot y) \qquad \frac{df}{dy}=-(x\cdot\sin(x\cdot y))$$

Integration of the first should give the terms containing *x*, that of the second should give the terms containing *y*:

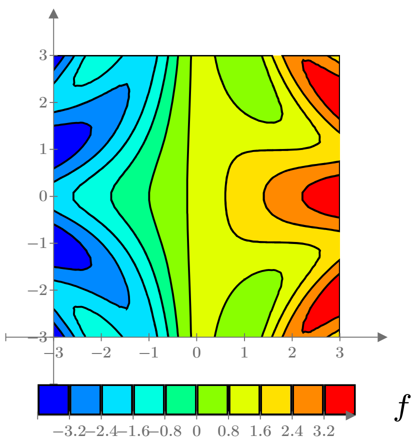
$$\int 1-y\cdot\sin(x\cdot y)\,dx \rightarrow x+\cos(x\cdot y)$$ terms containing *x*

$$\int -(x\cdot\sin(x\cdot y))\,dy \rightarrow \cos(x\cdot y)$$ terms containing *y*

The cosine term occurs in both integrals because it contains both an *x* and a *y*. The general solution is the *combination* of the two results (not the sum!).

$$f(x,y,C)=x+cos(x\cdot y)+C$$ as expected

A warning: you may get terms *not* containing *x* from the first integral, and terms *not* containing *y* from the second. Do not use these.



It is interesting to see how complicated the solutions of this simple-looking equation are.

The Integrating Factor

We take the differential equation from the previous paragraph, and divide it with some function (here simply *x*):

$$\left(\frac{1-y\cdot\sin(x\cdot y)}{x}\right)\cdot dx-\left(\frac{x\cdot\sin(x\cdot y)}{x}\right)\cdot dy=0 \qquad (1)$$

We then take the 'cross' derivatives of the terms in the new equation:

$$\frac{d}{dy}\frac{1-y\cdot\sin(x\cdot y)}{x} \rightarrow -\frac{\sin(x\cdot y)+x\cdot y\cdot\cos(x\cdot y)}{x}$$

$$\frac{d}{dx}-\left(\frac{x\cdot\sin(x\cdot y)}{x}\right) \rightarrow -(y\cdot\cos(x\cdot y))$$

These are not equal, so the new equation is no longer exact. However, if we multiply equation (1) with *x*, we get back the original, exact, equation. Here, *x* is an *integrating factor*: a factor which transforms the equation into one that can be solved. It can be proven that all first order, first degree equations have integrating factors. The problem is to find one of them. There is no general method for that.

3 First Order, Higher Degree Equations

These are equations containing a first order derivative to a degree higher than one. For example:

$$\left(\frac{dy}{dx}\right)^2 + x + y = 0$$

In the following dicussion, we will often use the substitution $p = \frac{dy}{dx}$

There are two groups of equations that can often be solved:
(1) The equation is the product of two equations of a lower degree.
(2) One of the variables *x* or *y* can be made explicit, so we can obtain:

(a) $y = f(p)$ or $x = f(p)$ (b) $x = f(y, p)$ or $y = f(x, p)$

These include the equations with the names of Clairaut and Lagrange.

Product of Equations

Let us take a simple example:

$$\left(\frac{dy}{dx}\right)^2 - \left(\frac{y}{x}\right)^2 = 0 \quad \text{which can be factored as} \quad \left(\frac{dy}{dx} + \frac{y}{x}\right) \cdot \left(\frac{dy}{dx} - \frac{y}{x}\right) = 0$$

There are two parts to this equation:

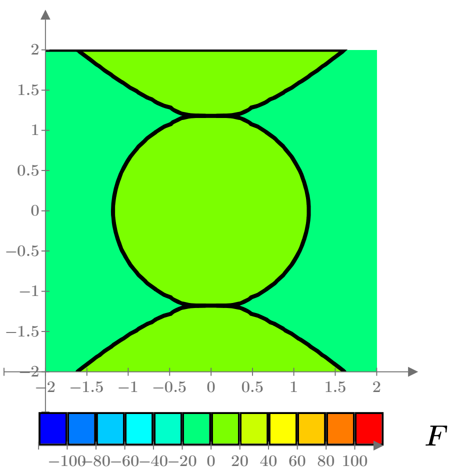
$$\left(\frac{dy}{dx} + \frac{y}{x}\right) = 0 \qquad y^2 = -x^2 + C \qquad y^2 + x^2 - C = 0$$

$$\left(\frac{dy}{dx} - \frac{y}{x}\right) = 0 \qquad y^2 = x^2 + C \qquad y^2 - x^2 - C = 0$$

(As the parts belong to the same solution, there is only one integration constant.) The general solution is:

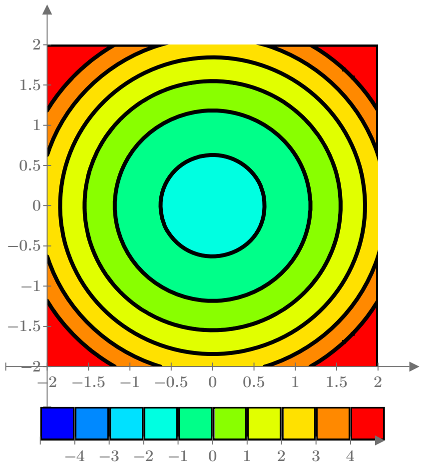
$$\left(y^2 + x^2 - C\right) \cdot \left(y^2 - x^2 - C\right) = 0$$

Try *C* = -1; 0; 1 and 2 to see the effect below. $C := 1.4$



The solution is the line at zero height. As you see, the solution is a combination of a circle and a hyperbola. These are the solutions of the two parts of the equation.

Below, I have made separate plots of the two solutions. The circles disappear for negative values of the integration constant; the hyperbola solutions switch from one branch to the other.



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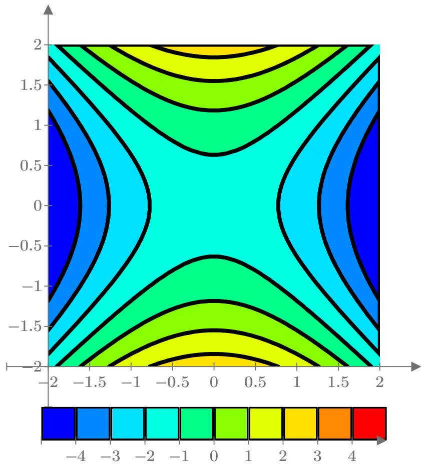

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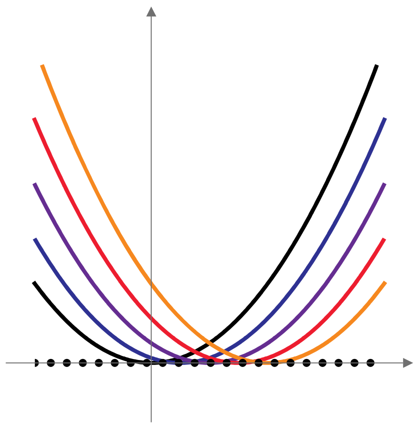
Explicit *y* = *f*(*p*)

Here we look at the following example:

$y = f(p) \qquad y = p^2 \qquad dy = p \cdot dx = f'(p) \cdot dp = 2 \cdot p \cdot dp$

This yields *x*: $\qquad dx = 2 \cdot dp \qquad x = 2 \cdot p + C \qquad p = \dfrac{x - C}{2}$



This parametric plot shows that the solutions are parabolas that shift along the *x*-axis with increasing *C*.

There is one more solution: $y = 0$
You can easily check this in the original equation. You cannot find it by giving *C* some value. This is a *singular* solution (which is not the same as a particular solution). The *x*-axis is tangent to all particular solutions

Explicit *x* = *f*(*p*)

These equations have the same solutions as those of the previous group, but with *x*- and *y*-axes exchanged. We will look at a different example:

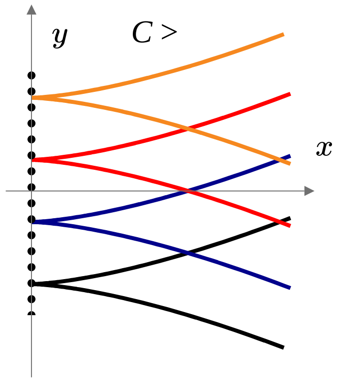
$$x = \left(\frac{dy}{dx}\right)^2 \qquad \text{or} \qquad x = p^2$$

This gives $\qquad dx = 2 \cdot p \cdot dp$

Also $\qquad dx = \dfrac{dy}{p} \qquad dy = p \cdot dx = 2 \cdot p^2 \cdot dp$

So $\qquad \int 2 \cdot p^2 \, dp \rightarrow \dfrac{2 \cdot p^3}{3} \qquad y = \dfrac{2}{3} \cdot p^3 + C$

The plot uses *p* as parameter: $x(p) = p^2$ $y(p) = \frac{2}{3} \cdot p^3 + C$



The solution is a cusp. Increasing *C* shifts the cusp upwards.

As you can see, all points of the *y*-axis obey the differential equation. You cannot see this in the general solution, so this is again a singular solution.

The contour plot of this example is not interesting.

Clairaut Equation

$\mathbf{clear}(x, y)$

The equation of Clairaut reads: $y = p \cdot x + f(p)$

I will consider the example: $y = p \cdot x + p^2$ (1)

We first differentiate $dy = p \cdot dx + x \cdot dp + 2 \cdot p \cdot dp$

Using $dy = p \cdot dx$ gives $0 = x \cdot dp + 2 \cdot p \cdot dp$

$(x + 2 \cdot p) \cdot dp = 0$

There are two solutions: $dp = 0$ (2) $x + 2 \cdot p = 0$ (3)

(2) gives $p = C$ (4)

(1) then gives the general solution: $y = C \cdot x + C^2$

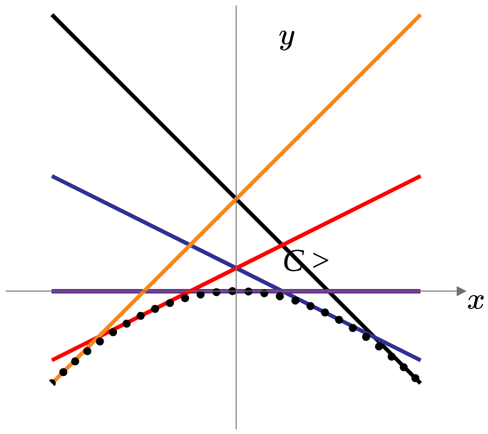
This is a set of straight lines with varying origin and slope.

Eliminating *p* from (1) and (3) gives an *envelope* of the solutions:

(3) $p = -\left(\frac{x}{2}\right)$ $y = p \cdot x + p^2$ $y = -\frac{x^2}{4}$

I have plotted several solutions below, together with the envelope:

$y(x, C) := C \cdot x + C^2$ $Y(x) := -\left(\frac{x^2}{4}\right)$



The particular solutions are tangent lines of the envelope. The envelope is a singular solution.

Note that there are no solutions inside the envelope.

Lagrange Equation

The equation of Lagrange has the form: $y = x \cdot f(p) + g(p)$

I will consider as example $y = x \cdot p^2 - p^2$ (1)

Differentiating gives $dy = 2 \cdot x \cdot p \cdot dp + p^2 \cdot dx - 2 \cdot p \cdot dp$

$$p \cdot dx = 2 \cdot x \cdot p \cdot dp + p^2 \cdot dx - 2 \cdot p \cdot dp$$

A first solution is $p = 0$

The second is $dx = 2 \cdot x \cdot dp + p \cdot dx - 2 \cdot dp$

This equation is linear in *x*. We solve it using the substitution:

$$x = u \cdot v \qquad dx = u \cdot dv + v \cdot du$$

$$u \cdot dv + v \cdot du = p \cdot (u \cdot dv + v \cdot du) + 2 \cdot u \cdot v \cdot dp - 2 \cdot dp$$

$$(1 - p) \cdot (u \cdot dv + v \cdot du) = 2 \cdot u \cdot v \cdot dp - 2 \cdot dp$$

$$(dv \cdot (1 - p) - 2 \cdot v \cdot dp) \cdot u + 2 \cdot dp - v \cdot du \cdot (p - 1) = 0$$

We choose *v* such that $dv \cdot (1 - p) - 2 \cdot v \cdot dp = 0$

$$\frac{dv}{v} = 2 \cdot \frac{dp}{1 - p}$$

$$ln(v) = -2 \cdot ln(p - 1)$$

$$v = \frac{1}{(p - 1)^2}$$

The remaining equation becomes

$$2 \cdot dp - v \cdot du \cdot (p - 1) = 0$$

$$du = 2 \cdot (p-1) \cdot dp$$

$$\int 2 \cdot (p-1)\, dp \rightarrow \frac{(2 \cdot p - 2)^2}{4} \qquad u = (p-1)^2 + C$$

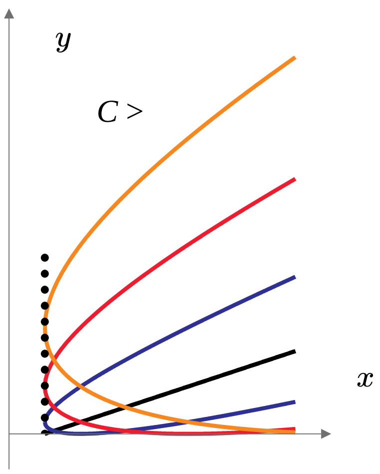
So the solution becomes

$$x = u \cdot v = 1 + \left(\frac{C}{(p-1)}\right)^2 \qquad y = (x-1) \cdot p^2$$

We can eliminate *p*. Each solution turns out to have two branches:

$$p = \frac{C}{\sqrt{x-1}} + 1 \qquad p = 1 - \frac{C}{\sqrt{x-1}}$$

$$y_1 = \left(\sqrt{x-1} + C\right)^2 \qquad y_2 = \left(\sqrt{x-1} - C\right)^2$$



The solution for $C = 0$ is a straight line. The others look like sheared parabolas.

The vertical line $x = 1$ is a singular solution.

4 Second Order, Linear Equations

Most equations with an order higher than one do not have a symbolic solution. Important exceptions are some of the linear equations. Here we look at second order linear equations with constant coefficients:

$$A \cdot \frac{d^2}{dx^2} y(x) + B \cdot \frac{d}{dx} y(x) + C \cdot y(x) = f(x) \qquad (1)$$

Linear equations have an important property: the sum of two solutions is also a solution. This is because

$$\frac{d^2}{dx^2}(f+g) = \frac{d^2}{dx^2} f + \frac{d^2}{dx^2} g \qquad \text{and} \qquad \frac{d}{dx}(f+g) = \frac{d}{dx} f + \frac{d}{dx} g$$

We first solve the 'homogeneous' equation:

$$A \cdot \frac{d^2}{dx^2} y(x) + B \cdot \frac{d}{dx} y(x) + C \cdot y(x) = 0 \qquad (2)$$

We then add a particular solution to include the effect of the right hand term. Even our simple equation cannot always be solved, but there are useful special cases where a solution can be obtained.

Homogeneoous Equation

The following function is a solution to the homogeneous equation:

$$y(x) = e^{k \cdot x} \qquad \frac{d}{dx} y(x) = k \cdot e^{k \cdot x} \qquad \frac{d^2}{dx^2} y(x) = k^2 \cdot e^{k \cdot x}$$

Inserting these in the homogeneous differential equation yields:

$$e^{k \cdot x} \cdot \left(A \cdot k^2 + B \cdot k + C\right) = 0$$

The quadratic equation between brackets is called the *characteristic equation*. It has two roots:

$$k_1 = \frac{-B + \sqrt{B^2 - 4 \cdot A \cdot C}}{2 \cdot A} \qquad k_2 = \frac{-B - \sqrt{B^2 - 4 \cdot A \cdot C}}{2 \cdot A} \qquad (3)$$

There are three cases:

(a) the roots are real and unequal,
(b) the roots are real, but equal, and
(c) the roots are complex.

Roots Real and Unequal

This is when $B^2 - 4 \cdot A \cdot C > 0$ The general solution of the homogeneous equation is:

$$y = C_1 \cdot exp(k_1 \cdot x) + C_2 \cdot exp(k_2 \cdot x) \qquad (4)$$

The solution has two integration constants, as might have been expected.

Roots Real and Equal

This is when $B^2 - 4 \cdot A \cdot C = 0$ The two roots have the same value: $k = \frac{B}{2 \cdot A}$
The reduced equation now has the general solution:

$$y = (C_1 + C_2 \cdot x) \cdot exp(k \cdot x) \qquad (5)$$

Roots Complex

Here the expression in the square root is negative: $B^2 - 4 \cdot A \cdot C < 0$

The solutions are now complex: $k_1 = a + b \cdot 1i$ $\quad k_2 = a - b \cdot 1i$

with $\quad a = \frac{-B}{2 \cdot A} \quad b = \frac{\sqrt{-(B^2 - 4 \cdot A \cdot C)}}{2 \cdot A}$

The reduced equation has the general solution:

$$y = C_1 \cdot \exp((a + b \cdot 1i) \cdot x) + C_2 \cdot \exp((a - b \cdot 1i) \cdot x)$$

The constants are often complex conjugates:

$$C_1 = a' + b' \cdot 1i \qquad C_2 = a' - b' \cdot 1i$$

The exponentials can then be replaced using the Euler equation:

$$e^{i \cdot z} = \cos(z) + i \cdot \sin(z)$$

This yields *periodic* solutions:

$$y = e^{a \cdot x} \cdot (C'_1 \cdot \cos(b \cdot x) + C'_2 \cdot \sin(b \cdot x)) \qquad (6)$$

Homogeneous Examples

The second order linear differential equation shows a wide range of behaviours. These are best studied using the equations (4), (5) and (6). (So not by varying the constants *A*, *B* and *C*.) Below, we look at a few examples.

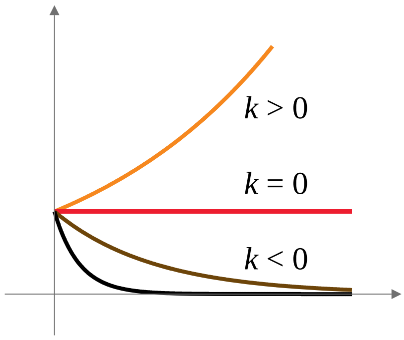
In the plots, I have left out indications along the axes. The constant considered is shown. The colours of the traces indicate how the values change: black < brown < red < orange.

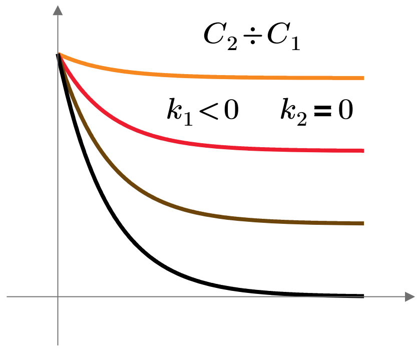
Growing and Shrinking

The solutions of equation (4) describe exponential growth and shrinkage. With four constants the equation is quite flexible.

$$y = C_1 \cdot exp(k_1 \cdot x) + C_2 \cdot exp(k_2 \cdot x) \qquad (4)$$

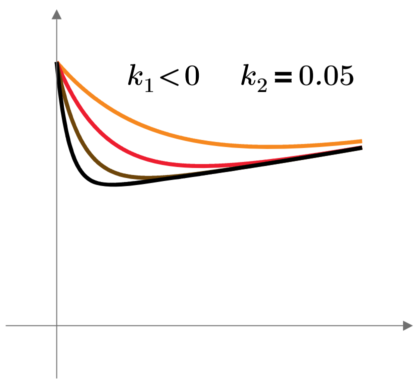
If one of the *C*'s is zero we have a single exponential as shown in the first figure. For $k < 0$ we get a falling curve; for $k > 0$ a rising one.





If the two exponentials have different *k*'s, a part of the solution will change rapidly, a part will not. This is most pronounced if one *k* is zero. If the other is negative, we can get 'levelling off'. The height of the 'plateau' depends on the ratio of the two integration constants.

If the first *k* has a negative value and the second a small positive value, we can get a rapid shrinkage, followed by slow growth:

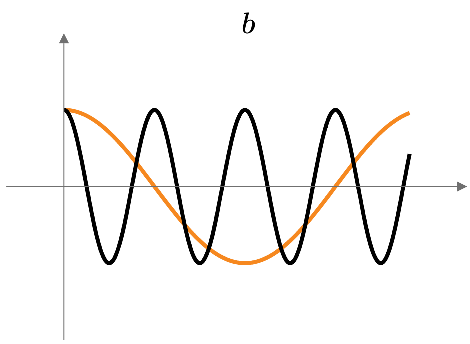


Oscillating

Equation (6) gives oscillating solutions. We first consider the case that the constant *a* in the exponential is zero and there is no 'damping':

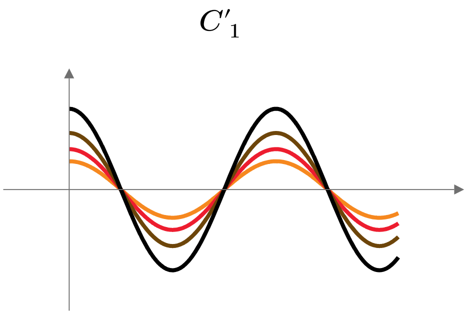
$$y = \left(C'_1 \cdot \cos\left(b \cdot x\right) + C'_2 \cdot \sin\left(b \cdot x\right)\right) \qquad (6\ a)$$

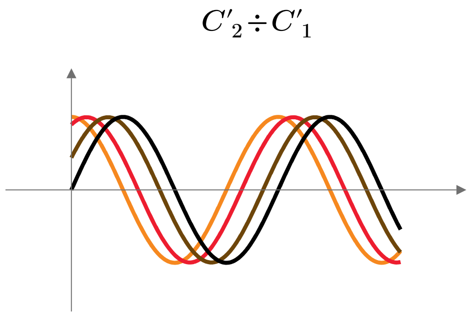
The frequency is determined by *b*: $b = \sqrt{-\left(\frac{C}{A}\right)}$



I have only plotted one high and one low value.

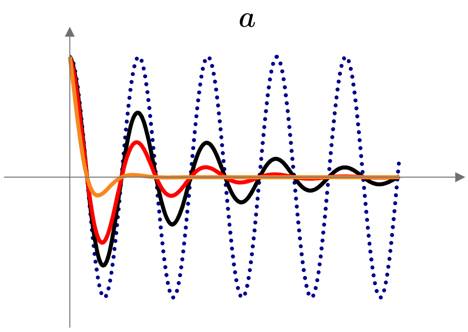
The constants before the cosine and sine determine the amplitude and phase, as you see in the following figures.





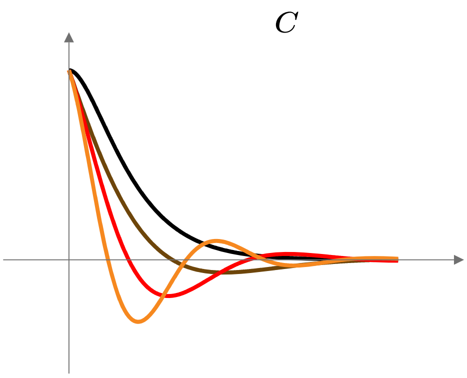
Damping is governed by the parameter *a*; the more negative this is, the larger the damping.

$$y = e^{a \cdot x} \cdot C'_1 \cdot \cos(b \cdot x) \qquad a = \frac{-B}{2 \cdot A} \qquad (6\ b)$$



The Transition

We finish with a look at the transition between the exponential and oscillating regimes. This is not as abrupt as you might think. At the transition, the frequency *b* is zero; just inside the oscillating regime the oscillations will be slow. If there is any damping, you may not notice the oscillation.



The black line is that of the transition (equation 5). The others are for increasing values of the constant *C* in the differential equation (so for less and less damping).

The Particular Solution

We now consider the effect of the right side of our equation (the *forcing function*):

$$A \cdot \frac{d^2}{dx^2} y(x) + B \cdot \frac{d}{dx} y(x) + C \cdot y(x) = f(x)$$

We 'only' need to add a particular solution of the whole equation. The particular solution does not contain integration constants. It can only be determined for certain kinds of functions and their combinations, such as:

function $f(x)$	particular solution
x	$y_P(x) = b \cdot x + c$
x^2	$y_P(x) = a \cdot x^2 + b \cdot x + c$
$e^{k \cdot x}$	$y_P(x) = a \cdot e^{k \cdot x}$
$\sin(k \cdot x)$	$y_P(x) = a \cdot sin(k \cdot x) + b \cdot cos(k \cdot x)$

The constants are determined by inserting the particular solution into the differential equation. As an example we consider

$$\frac{d^2}{dx^2} y(x) - 5 \cdot \frac{d}{dx} y(x) + 6 \cdot y(x) = x$$

The characteristic equation has the solutions:

$$k^2 - 5 \cdot k + 6 \xrightarrow{solve, k} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So the solution of the homogeneous equation is

$$y_H = C_1 \cdot exp(2 \cdot x) + C_2 \cdot exp(3 \cdot x)$$

We try the particular solution:

$$y_P(x) = b \cdot x + c \qquad \frac{d}{dx}(b \cdot x + c) = b \qquad \frac{d^2}{dx^2}(b \cdot x + c) = 0$$

Inserting this in the differential equation yields

$$0 - 5 \cdot b + 6 \cdot (b \cdot x + c) = x \qquad (6 \cdot b - 1) \cdot x + (6 \cdot c - 5 \cdot b) = 0$$

This has to be true for all *x*, so $b = 1 \div 6$ $c = 5 \div 36$

The general solution of the equation is then

$$y = y_H + y_P = C_1 \cdot exp(2 \cdot x) + C_2 \cdot exp(3 \cdot x) + \frac{x}{6} + \frac{5}{36}$$

This procedure can require a lot of work.

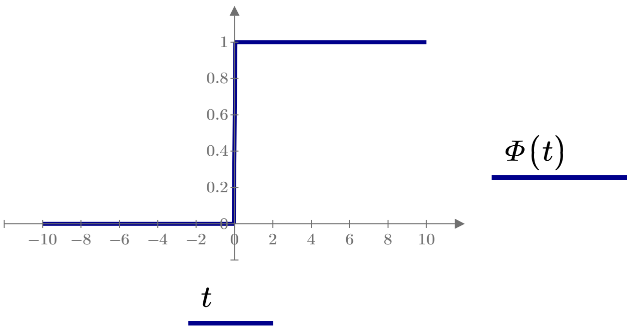
5 Laplace Transforms

With the method of lesson 4 we can only obtain solutions to linear equations for some simple forcing functions. With Laplace transforms we can handle more interesting cases - and in a simpler manner.

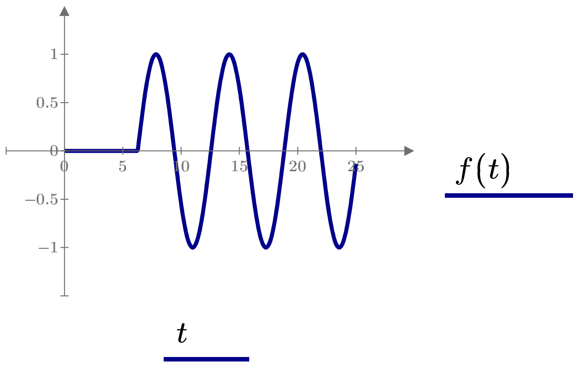
These transforms are mostly used in dynamics - where the independent variable is time. So you will be seeing a *t* where we had an *x* in earlier lessons. Before we look at the transform, I will first introduce two functions that are useful for describing abrupt changes.

Abrupt Changes

The *step* function (which is also known as the Heaviside function) gives a unit step at *t* = 0:



You can use it to construct other functions: $f(t) := \Phi(t - 2 \cdot \pi) \cdot \sin(t)$



The *impulse* function (also known as the Dirac delta) gives an infinite 'bang' at *t* = 0, but is zero elsewhere. You cannot plot it. However, you can check that its integral has a value of one:

$$\int_{-\infty}^{\infty} \Delta(t)\, dt \rightarrow 1$$

The two functions are related; the derivative of the step is an impulse:

$$\frac{d}{dt} \Phi(t) \rightarrow \Delta(t)$$

Using Laplace

In our differential equations, the independent variable will be *t*. The Laplace transform changes all expressions into functions of a different variable *s*. We will see later how this is done, but must say that it is not a simple substitution.

The important property of the Laplace transform is that it changes derivatives into algebraic expressions in *s*. We can then solve these equations. With an inverse transform we then get the solution as a function of *t*. I will show this with the following differential equation:

$$y''(t)+4\cdot y'(t)+20\cdot y(t)=\Delta(t-1)$$

The forcing function is an impulse at *t* = 1. Also here, you need to specify initial conditions. It is simplest if you take these at *t* = 0:

$$y(0)=0 \qquad y'(0)=0$$

The Laplace transform of the first term in the equation is:

$$y''(t)\xrightarrow{laplace} s^2\cdot laplace(y(t),t,s)-y'(0)-s\cdot y(0)$$

Inserting the initial conditions gives $s^2\cdot laplace(y(t))$

Similarly, the second term $4\cdot y'(t)$ gives $4\cdot s\cdot laplace(y(t))$

The third term $20\cdot y(t)$ gives $20\cdot laplace(y(t))$

and finally the forcing term gives $\Delta(t-1)\xrightarrow{laplace} e^{-s}$

With the initial conditions the transformed differential equation becomes:

$$\left(s^2+4\cdot s+20\right)\cdot laplace(y(t))=e^{-s}$$

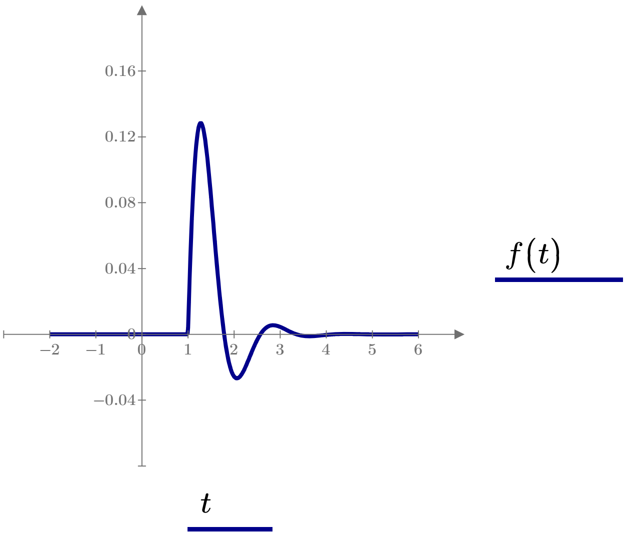
$$laplace(y(t))=\frac{e^{-s}}{\left(s^2+4\cdot s+20\right)}$$

Inverting the transformation gives the symbolic result:

$$\frac{e^{-s}}{\left(s^2+4\cdot s+20\right)}\xrightarrow{invlaplace}\frac{e^{2-2\cdot t}\cdot\sin(4\cdot t-4)\cdot\Phi(t-1)}{4}$$

$$f(t):=\frac{e^{2-2\cdot t}\cdot\sin(4\cdot t-4)\cdot\Phi(t-1)}{4}$$

That is all there is to it! I have plotted the result below.



The impulse gives the system an amplitude, which then dies out. Note that I have chosen to have the impulse *not* at *t* = 0. If you do that, you will not be able to specify the initial condition.

$$\mathbf{clear}(y, t)$$

As a second example, I choose a first order linear equation, but with a discontinuous sine as forcing function:

$$y'(t) + y(t) = f(t) \qquad y(0) = 0 \qquad f(t) := \Phi(t - 2\cdot\pi)\cdot\sin(t)$$

The Laplace transforms of the three terms are:

$$y'(t) \qquad s\cdot laplace(y(t))$$

$$y(t) \qquad laplace(y(t))$$

$$f(t) \qquad \frac{e^{-2\cdot\pi\cdot s}}{s^2+1}$$

So the transformed equation becomes

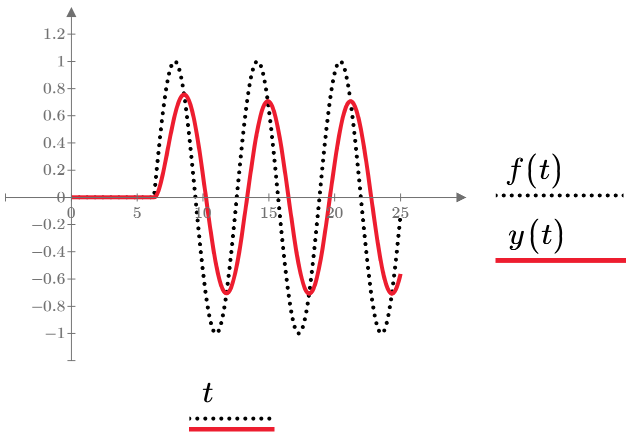
$$(s+1)\cdot laplace(y(t)) = \frac{e^{-2\cdot\pi\cdot s}}{s^2+1} \qquad laplace(y(t)) = \frac{e^{-2\cdot\pi\cdot s}}{(1+s)\cdot(s^2+1)}$$

The solution is:

$$\frac{e^{-2\cdot\pi\cdot s}}{(1+s)\cdot(s^2+1)} \xrightarrow{invlaplace} \Phi(t-2\cdot\pi)\cdot\left(\frac{e^{2\cdot\pi-t}}{2} - \frac{\cos(t)}{2} + \frac{\sin(t)}{2}\right)$$

$$y(t) := \Phi(t-2\cdot\pi)\cdot\left(\frac{e^{2\cdot\pi-t}}{2} - \frac{\cos(t)}{2} + \frac{\sin(t)}{2}\right)$$

I have plotted this below, together with the forcing function.



After the start, the system follows the forcing function with a phase lag.

The Transform

The Laplace transform is an integral transform. It is defined as:

$$laplace\left(y(t)\right) = \int_0^{\infty} y(t) \cdot e^{-s \cdot t}\, dt$$

We have already seen the transforms of a few functions. Here are two others. I have included the integral calculations to show that they really work. (You do often have to restrict the range of integration to get the same result as that of the keyword. The keyword is simpler and much faster than the integral.)

$$\Phi(t) \xrightarrow{laplace} \frac{1}{s} \qquad \int_0^{\infty} \Phi(t) \cdot e^{-s \cdot t}\, dt \xrightarrow[\;]{\substack{assume\\ s>0}} \frac{1}{s} \qquad \frac{1}{s} \xrightarrow{invlaplace} 1$$

$$\cos(t) \xrightarrow{laplace} \frac{s}{s^2+1} \qquad \int_0^{\infty} \cos(t) \cdot e^{-s \cdot t}\, dt \xrightarrow[\;]{\substack{assume\\ s>0}} \frac{s}{s^2+1}$$

$$\frac{s}{s^2+1} \xrightarrow{invlaplace} \cos(t)$$

The inverse transformation is not a simple function. However, you do not deed to worry about that as Mathcad handles it for you.

On the next page I have constructed a table of transforms such as you might find in a course book in 'Operational Mathematics'. In Mathcad you can construct this in perhaps an hour.

Table of Laplace Transforms

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$e^{a \cdot t}$	$\frac{1}{s-a}$
t^3	$\frac{6}{s^4}$
$t^3 \cdot e^{a \cdot t}$	$\frac{6}{(a-s)^4}$
$sin(k \cdot t)$	$\frac{k}{k^2+s^2}$
$cos(k \cdot t)$	$\frac{s}{k^2+s^2}$
$sinh(k \cdot t)$	$-\frac{k}{k^2-s^2}$
$e^{-a \cdot t} \cdot sin(k \cdot t)$	$\frac{k}{a^2+2 \cdot a \cdot s+k^2+s^2}$
$e^{-a \cdot t} \cdot cos(k \cdot t)$	$\frac{a+s}{a^2+2 \cdot a \cdot s+k^2+s^2}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2 \cdot s^{\frac{3}{2}}}$
$\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$
$\Phi(t-k)$	$\frac{e^{-2 \cdot s}}{s}$
$e^{a \cdot t}-e^{b \cdot t}$	$\frac{a-b}{(a-s) \cdot (b-s)}$
$\frac{1}{a} \cdot sin(a \cdot t)-\frac{1}{b} \cdot sin(b \cdot t)$	$-\frac{a^2-b^2}{(a^2+s^2) \cdot (b^2+s^2)}$
$cos(a \cdot t)-cos(b \cdot t)$	$-\frac{s \cdot (a^2-b^2)}{(a^2+s^2) \cdot (b^2+s^2)}$

6 Numerical Methods

These are often used to solve practical problems. Mathcad has powerful numerical methods. These can do things that symbolic techniques cannot, but you will have to learn to use them.

Introduction

As an introduction, we look at the Euler method. This is simple - not accurate, but easy to understand. The equation that we are going to solve is:

$$y'(x) = -k \cdot y \qquad k := 1$$

We approximate the equation as: $\frac{\Delta y}{\Delta x} = -k \cdot y$ or $\Delta y = -k \cdot y \cdot \Delta x$

This will be solved over the interval $x_S < x < x_E$

The starting and end values are $x_S := 0$ $x_E := 3$

With this first order equation, we need a single initial condition:

$y(x_S) = 2$ that I shall write as $x_0 := 0$ $y_0 := 2$

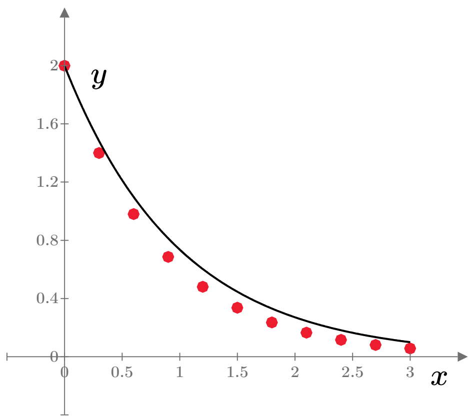
We solve the equation in a number of steps:

steps $n := 10$ increment $\Delta x := \frac{x_E - x_S}{n}$ $\Delta x = 0.3$

The *x*-positions become $i := 1 .. n$ $x_i := x_{i-1} + \Delta x$

and the estimates of *y* $y_i := y_{i-1} - k \cdot y_{i-1} \cdot \Delta x$

I have plotted the calculated values below as red points. Also shown is the exact solution of the equation. You can see the trend, but this numerical solution is not accurate.



Mathcad contains quite a collection of numerical routines for differential equations, as you can see under 'Functions'. You will not find the Euler method there: it is not good enough.

The one you will see in a moment is **odesolve**. It makes use of five of the other routines that you see in the list: Adams, BDF, rkfixed, Rkadapt and Radau. It chooses the best combination, depending on the behaviour of your equation. All these start with a grid and work from point to point (as does Euler). However, they are faster and much more accurate.

Using Odesolve

$\mathbf{clear}(x, y)$

As an example I will use the linear second order differential equation in the solve block below. It is the same one as in the previous lesson. Ahead of the solve block are the four parameters in the differential equation. Also given are the values of the independent variable (here *x*) at the start and end of the range to be used. There are three regions in the solve block:

(1) Guess Values
(2) Constraints
(3) Solver.

For differential equations, guess values are not used. For our second order equation we need two initial conditions in the constraints region: one initial value of the dependent variable *y*, and one for its derivative. The differential equation is given using prime notation and should be clear. The solver is **odesolve( )**; between the brackets it contains the dependent variable and the end value of the dependent variable.

equation parameters $A := 1$ $B := 1$ $C := 3$ $D := 0$

start and end $x_S := 0$ $x_E := 10$

Guess Values

region for guess values (not used)

Constraints

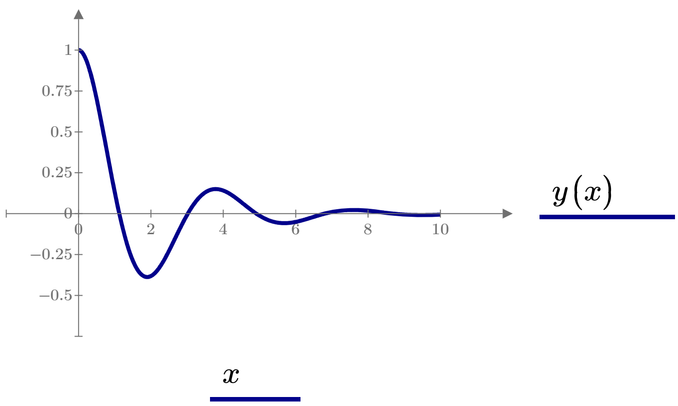
$y(x_S) = 1 \quad y'(x_S) = 0$

$A \cdot y''(x) + B \cdot y'(x) + C \cdot y(x) + D = 0$

Solver

$y := \mathbf{odesolve}(y(x), x_E)$

The output is a damped oscillation. It is the *particular solution* for the given initial values.



Capabilities

You can do many things with numerical techniques that are difficult or impossible with others:

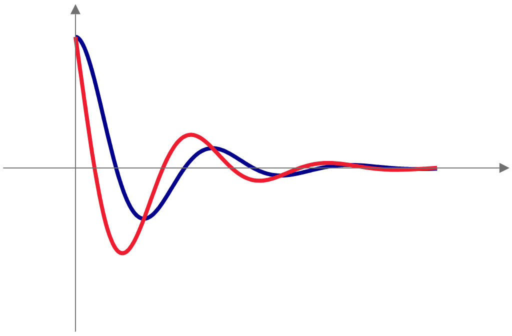
(a) solve boundary value problems
(b) use arbitrary forcing functions
(c) have coefficients that are functions of *x*
(d) include terms that are non-linear in *y*(*x*).

I will show these using variations of our first example. I have hidden the solve blocks, as they are almost the same as that of the example. After each calculation you will see a plot with the original result in black and the modified one in red.

Boundary Value Problems

So far, we have only considered initial value problems. Here, the value of the function and derivatives are specified at the start of the calculation. For equations with an order higher than one, you can also specify an end condition. Below is such a boundary value problem.

$$y(x_S) = 1 \qquad y(x_E) = 0 \qquad A \cdot y''(x) + B \cdot y'(x) + C \cdot y(x) + D = 0$$

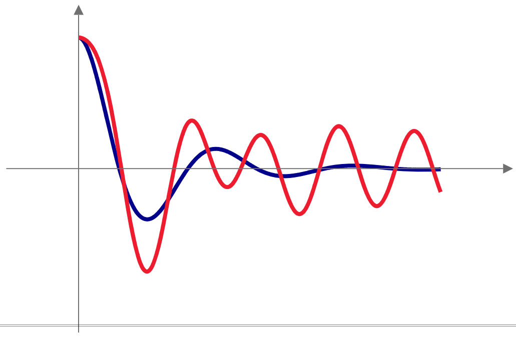


In the example here, the result is sensitive to the choice of the final condition. I have chosen it such that the two results do not differ greatly.

The Forcing Function

In the numerical technique, you can use arbitrary forcing functions. (However, do not try functions with infinite gradients such as the step and impulse functions.) In the example, I have used a simple cosine.

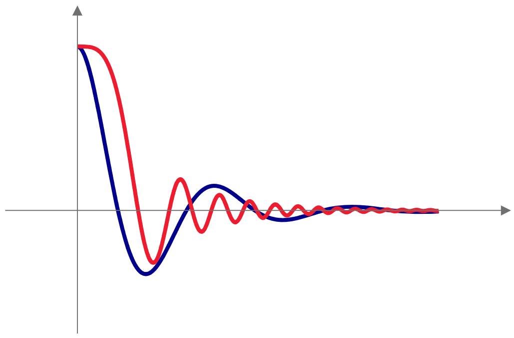
$$A \cdot y''(x) + B \cdot y'(x) + C \cdot y(x) + D = \cos(3 \cdot x)$$



Variable Coefficients

The numerical solution can handle variable coefficients. In the equation below, I have inserted an *x* squared:

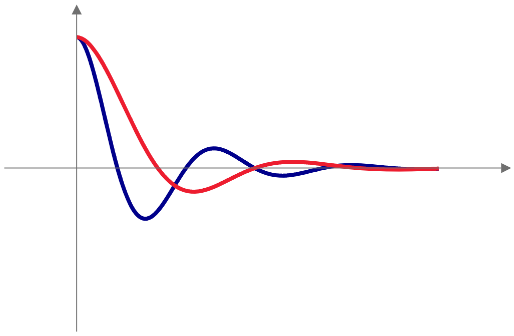
$$A \cdot y''(x) + B \cdot y'(x) + C \cdot x^2 \cdot y(x) + D = 0$$



Non-linear Equations

You can also include non-linear coefficients and arbitrary forcing functions. (However, non-linear equations often do not give a solution.)

$$A \cdot y''(x) + B \cdot y'(x) + C \cdot \frac{y(x)}{y(x)+2} + D = 0$$



Rules are Rules

The Mathcad routine for solving differential equations numerically is one of the few parts of the interface that is not friendly. You have to follow certain not-so-obvious rules exactly. So reckon on needing patience when you begin. (It does help to copy a file that works, then to modify it.) In this discussion of the rules, *x* is the independent variable, *y* the dependent one. (You can use any set of variables.)

Parameters

Parameters are given as numerical values *before* the solve block. You cannot use assignments inside the solve block. You can use numbers in the initial conditions or in the equation.

Start and End
The start and end values of the independent variable *x* are specified at two different points. The start value (often 0) is given in all initial conditions. The end value is specified in the solver, but can also be used in a boundary condition. The end value can be either larger or smaller than the start value. In plots you only see results in the range specified.

Dependent Variable
Throughout the solve block, the dependent variable *y* is written as a function *y*(*x*) of the independent variable *x*. There is *one exception*: the result of the solver is assigned to *y*, **not** to *y*(*x*). However, if you want to use the result in a plot, you must again use *y*(*x*). The result is not a true function, but a set of two vectors of calculated points. However, you never see these.

Derivatives
The derivatives in the equation can be written in two ways:

(1) with the derivative operator $\frac{d}{dx}y(x)$ $\frac{d^2}{dx^2}y(x)$

(2) with the prime operator $y'(x)$ $y''(x)$

The prime operator is typed using [Ctrl]['], *not* using the prime. For derivatives in initial conditions, you can only use prime notation.

Equalities
All equalities in the solve block are Boolean equalities =, typed using [Ctrl][=]. *There is one exception*: the assignment := in the solver, typed with [Ctrl][:].

Errors do give error messages. Unfortunately, these are often difficult to understand.

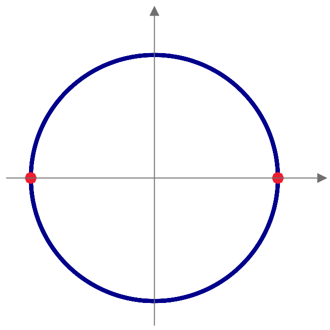
Does Not Work With...
There are many equations that do not have a symbolic solution, but which can be solved numerically. However, there are also equations where it is the other way around. I will illustrate that with one example from lesson 2 - this illustrates some of the limitations of numerical methods.

The Circle
The example is the equation: $y'(x) = \frac{y}{x}$ or $y \cdot dy = x \cdot dx$

The solution to this is a series of circles around the origin. You would expect to get a circle with radius one by specifying the initial condition:

$$y(0) = 1$$

If you try this numerically, you do not get an answer. To see why, let us plot the circle.



The first thing to note is that the function has a limited domain:

$$-1 \leq x \leq 1 \qquad -1 \leq y \leq 1$$

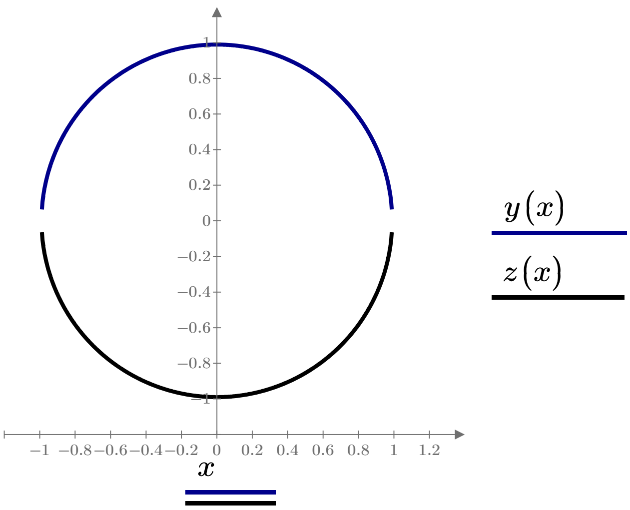
Your start and endpoints must lie in this domain. The second thing is that the function is double-valued. For each *x*, there are two *y*'s. Our method does not give these automatically.

The third thing to note is the infinite slope of the function at the points (0,1) and (0,-1). Numerical methods cannot handle such points. I have managed to calculate the circle, but in a roundabout way. I have calculated the two branches of the circle separately. Also, I have avoided the two red points:

$$y'(x) = \frac{-x}{y(x)} \qquad z'(x) = \frac{-x}{z(x)}$$

$$-0.99 < x < 0.99$$

$$y(-0.99) = 10^{-3} \qquad z(-0.99) = -10^{-3}$$



In this example it would have been more elegant to switch to polar co-ordinates, but that does not always solve problems such as here either.