

Lecture 3 - Solving for the Dynamic Response

Friday, January 11, 2013

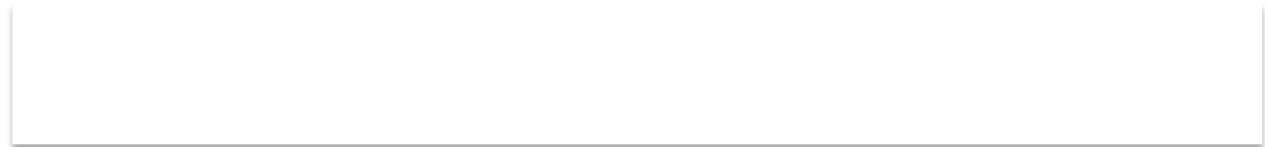
Today's Objectives

1. review the solution of homogeneous differential equations in the time domain (using assumed solution)
2. solve differential equations using the Laplace transform

Reading: FPE Section 3.1, Appendix A.1.1

Solving for the dynamic response

The models of systems we obtain from the modeling techniques described in the last lecture can be put in a form:



These are linear, time-invariant (LTI), constant-coefficient ODEs describing a single input, single output (SISO) system. While not everything can be put into this form, many systems can. There are so many analytical tools available for systems of this form that it often makes sense to try to fit the system into this form as a starting point.

In an ODE class, these equations are solved for two solutions:

Homogeneous

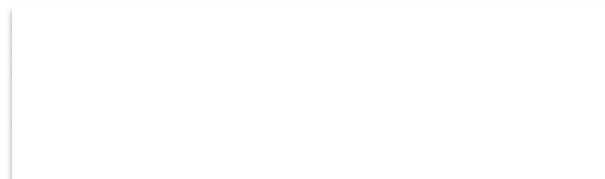
- input = 0
- free response
- natural response

Particular

- depends on input
- forced response

1 Time domain solutions

The homogeneous solution is straightforward:



⇒ substitute this assumed solution into the ODE and then solve for A and s .

Some figures in this document ©2010 Pearson (from the textbook Feedback Control of Dynamic Systems, 6th Ed.)

Mass-spring-damper example

There are two solutions:

The complete solution can be written as:

How do we get values for A_1 and A_2 ? You need to define the initial conditions:

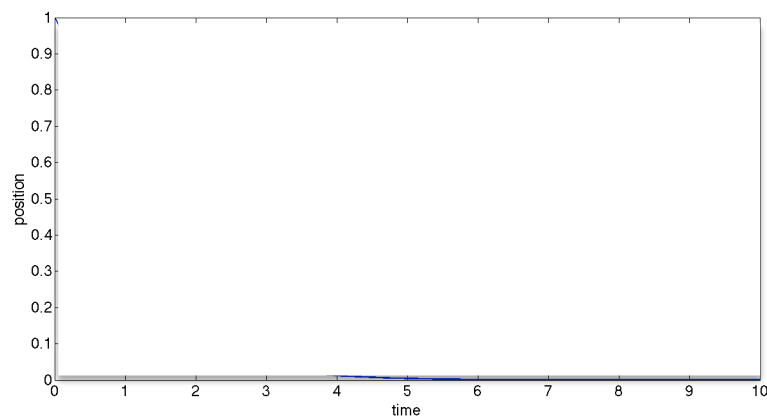
We have two equations and two unknowns, so you can solve for A_1 and A_2 .

What do these solutions look like?

Here are two cases:

(a) If $b^2 > 4mk$, both s_1 and s_2 are negative and real

Here is a plot of $y = A_1 e^{s_1 t} + A_2 e^{s_2 t}$, where $s_1, s_2 < 0$ and real:

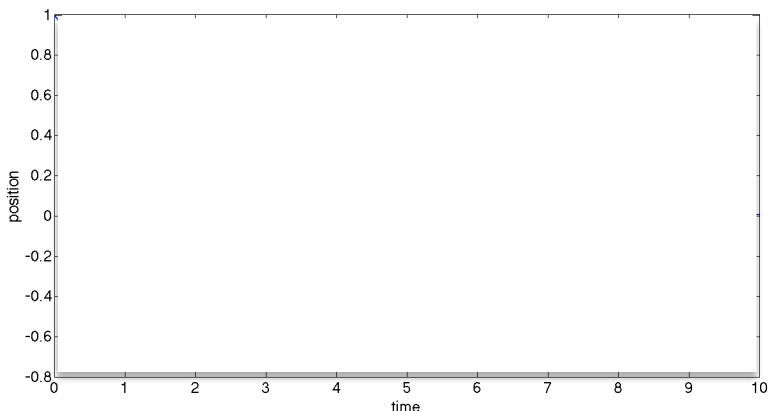


```
m = 0.1;
b = 1;
k = 1;
ydot0 = 0;
y0 = 1;
s1 = (-b + sqrt(b^2-4*m*k))/(2*m);
s2 = (-b - sqrt(b^2-4*m*k))/(2*m);
A1 = (ydot0-y0*s2)/(s1-s2);
A2 = (ydot0-y0*s1)/(s2-s1);
t = 0:0.1:10;
y = A1*exp(s1*t) + A2*exp(s2*t);
figure
plot(t,y)
xlabel('time')
ylabel('position')
```

(b) If $b^2 < 4mk$, s_1 and s_2 are complex (in fact, complex conjugates)

These will oscillate, since Euler's formula says that:

Here is a plot of $y = A_1 e^{s_1 t} + A_2 e^{s_2 t}$, where s_1, s_2 are complex with negative real parts:



```
m = 0.1;
b = 0.1;
k = 1;
ydot0 = 0;
y0 = 1;
s1 = (-b + sqrt(b^2-4*m*k))/(2*m);
s2 = (-b - sqrt(b^2-4*m*k))/(2*m);
A1 = (ydot0-y0*s2)/(s1-s2);
A2 = (ydot0-y0*s1)/(s2-s1);
t = 0:0.1:10;
y = A1*exp(s1*t) + A2*exp(s2*t);
figure
plot(t,y)
xlabel('time')
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```

- Is there another possible case?
- Solutions to equations of this form are always real roots or complex conjugate pairs. (Why?)

Now imagine there is a force applied to the mass, so the equation of motion becomes $m\ddot{y} + b\dot{y} + ky = f$. There are several ways to find the particular response to such a system using time-domain analysis, and these depend on the form of f . We will not review these techniques in this class because there is another solution technique that handles the homogeneous and particular solutions at once. It also provides greater insight about system structure. This is the Laplace transform.

2 Laplace transforms

A Laplace transform transforms:

- 1.
- 2.
- 3.

The solution process for ODEs looks like:

Definition of the Laplace transform

Given a function $f(t)$,

s is a complex number:

Superposition

It naturally follows that:

Exponential

Since the exponential is such an important building block in our ODEs, it makes sense to look in more detail at the Laplace transform of the exponential:

Step function

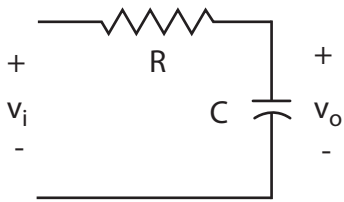
Notice that the exponential converges to a step as $\alpha \rightarrow 0$.

Differentiation

Here is the result:

This is a great example of integration by parts:

RC circuit example



We have just developed the *transfer function* for the RC circuit. Now we will solve for $v_o(t)$.

Particular solution for a step input $V_i(s) = \frac{A}{s}$: