ENGR 105: Feedback Control Design
Winter 2013

Lecture 2 - System Modeling

Wednesday, January 9, 2013

Today's Objectives

1. define basic elements and laws of electrical circuits
2. describe ideal op-amps
3. learn to draw free-body diagrams for mechanical systems
4. provide electromechanical system example

Reading: FPE Sections 2.1, 2.2, 2.3

System modeling

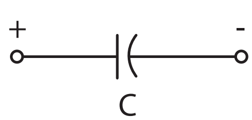
The analysis and design tools in this class apply to dynamic system models that can represent a range of different physical systems. While developing models of mechanical or electrical systems can be a course by itself, there are a few basic concepts needed for this class, which we will review now:

Electrical systems	Mechanical systems
• definitions of elements	• free body diagrams
• Kirchoff's laws	• $f = ma$
• golden rules of op-amps	

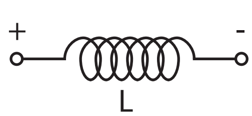
1 Electrical system elements and Kirchoff's laws

resistor 

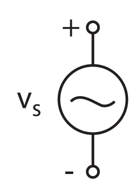
$v = iR$

capacitor 

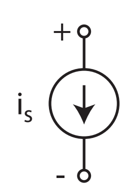
$i = C\frac{dv}{dt}$

inductor 

$v = L\frac{di}{dt}$

voltage source 

$v = v_s$

current source 

$i = i_s$

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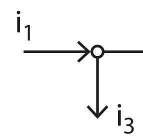
Each capacitor or inductor adds one derivative to the model and thus adds one state (i.e., variable whose value must be known to completely describe the behavior of the system) to the model. Resistors do not involve derivatives and thus do not add states to the model.

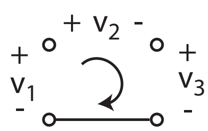
Kirchoff's laws

current law



$i_1 = i_2 + i_3$ The sum of currents flowing into a node is zero.

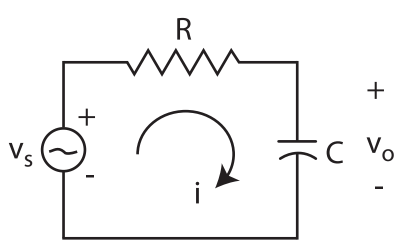
voltage law



$v_1 = v_2 + v_3$ The sum of voltages taken around a closed path is zero.

Basic circuit example

Here we use these rules for a simple RC circuit:



$v_s - iR = v_o$

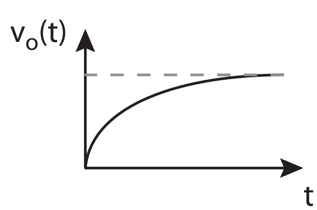
$i = C\frac{dv_o}{dt}$

$v_s - RC\frac{dv_o}{dt} = v_o$

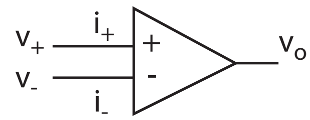
$\frac{dv_o}{dt} = \frac{1}{RC}(v_s - v_o)$

For zero initial conditions, this has the solution: $v_o = (1 - e^{\frac{-t}{RC}})v_s$

The first order response looks like:



2 Ideal op-amps

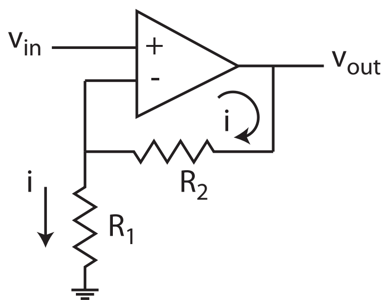


Ideal op-amps satisfy two rules:

1. v_o does whatever it takes to make $v_+ = v_-$
2. $i_+ = i_- = 0$, so no current is drawn

With these two rules, Kirchoff's laws, and the definition of the elements, equations can be derived for basic op-amp circuits.

Example 1



$$v_- = v_{\text{out}} - iR_2 = iR_1 = v_{\text{in}}$$

$$v_{\text{in}} = v_{\text{out}} - iR_2$$

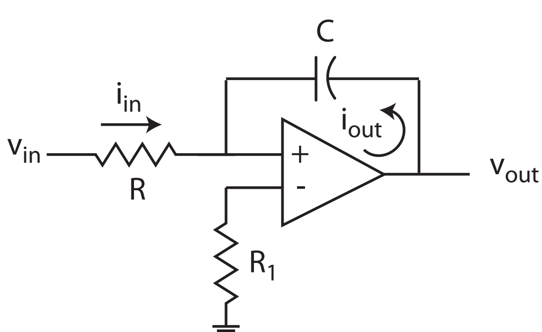
$$i = \frac{v_{\text{in}} - v_{\text{out}}}{R_2}$$

$$v_{\text{in}} = iR_1 = \frac{R_1}{R_2}(v_{\text{in}} - v_{\text{out}})$$

$$v_{\text{out}} = v_{\text{in}}\left(1 + \frac{R_2}{R_1}\right)$$

This is just a static gain. There are no dynamics in this system because no derivatives are involved (no capacitors or inductors).

Example 2



$$i_{\text{in}} = -i_{\text{out}}$$

$$\frac{v_{\text{in}}}{R} = -C\frac{dv_{\text{out}}}{dt}$$

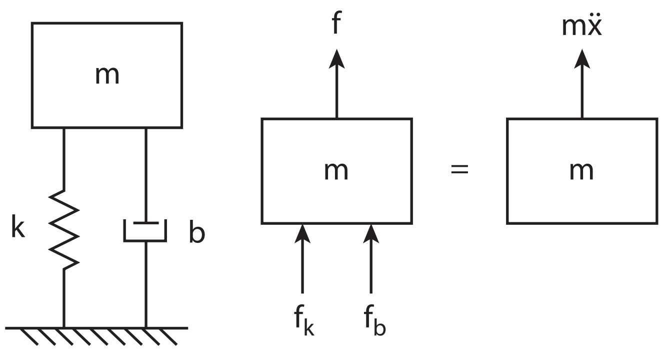
$$\frac{dv_{\text{out}}}{dt} = -\frac{1}{RC}v_{\text{in}}$$

For zero initial conditions,, this has the solution: $v_{\text{out}} = -\frac{1}{RC}\int_0^t v_{\text{in}}(\tau)d\tau$

This is an integrator circuit. It does have dynamics due to the addition of the capacitor.

3 Free body diagrams for mechanical systems

The key to developing equations of motion for mechanical systems is to draw a free body diagram for each individual mass in the system, which shows all of the forces acting on that mass. The equations of motion follow from $f = ma$. (We won't do complicated 3D dynamics in this class, but we will examine systems that have multiple degrees of freedom.)



spring: $f_k = -kx$

damper: $f_b = -b\dot{x}$

$\sum f = ma = m\ddot{x}$

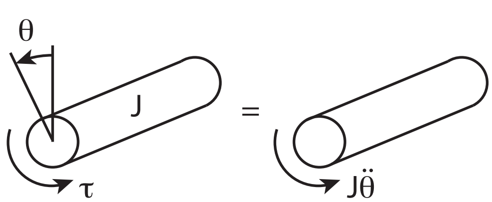
$f + f_k + f_b = m\ddot{x}$

$f - kx - b\dot{x} = m\ddot{x}$

$m\ddot{x} + b\dot{x} + kx = f$

note: can include gravity on right-hand-side, or just include in kx for rest position

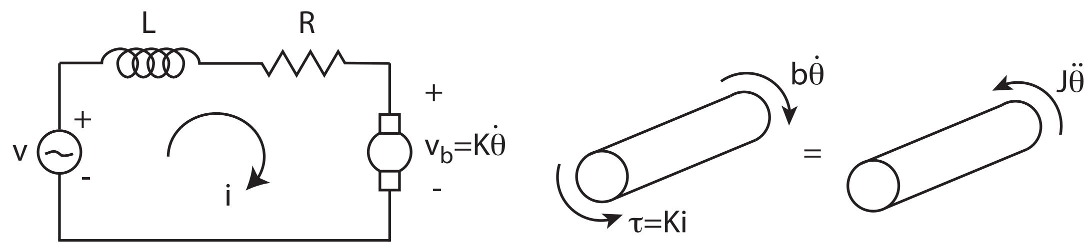
The rotational form is pretty much the same. Just think in terms of torque and inertia instead of force and mass. Angular acceleration replaces linear acceleration.



$\sum \tau = J\ddot{\theta}$

4 Electromechanical system example

Combining mechanical and electrical modeling, we can develop a simple motor model:



$$v - L\frac{di}{dt} - iR - K\dot{\theta} = 0 \qquad\qquad J\ddot{\theta} = \tau - b\dot{\theta}$$

$$L\frac{di}{dt} + iR + K\dot{\theta} = v \qquad\qquad J\ddot{\theta} + b\dot{\theta} = Ki$$

These two equations define the dynamics of the motor. Often L is small relative to R and K. In this case, the electrical dynamics are approximately:

$iR + K\dot{\theta} = v$

So the two equations can be combined as: $\quad J\ddot{\theta} + \left(b + \frac{K^2}{R}\right)\dot{\theta} = \frac{K}{R}v$

In this case, we have a second-order system (two derivatives). If we are interested in positions, we get two states, or two derivatives, from each "mass" in the system.

If we only care about angular velocity, define $\omega = \dot{\theta}$

$J\dot{\omega} + \left(b + \frac{K^2}{R}\right)\omega = \frac{K}{R}v$

This is a first-order system.