ENGR 105: Feedback Control Design
Winter 2013

Lecture 3 - Solving for Dynamic Response

Friday, January 11, 2013

Today's Objectives

1. review the solution of homogeneous differential equations in the time domain (using assumed solution)
2. solve differential equations using the Laplace transform

Reading: FPE Section 3.1, Appendix A.1.1

Solving for the dynamic response

The models of systems we obtain from the modeling techniques described in the last lecture cane put in a form:

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_1 \frac{d^m u}{dt^m} + b_2 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_{m+1} u$$

These are linear, time-invariant (LTI), constant-coefficient ODEs describing a single input, single output (SISO) system. While not everything can be put into this form, many systems can. There are so many analytical tools available for systems of this form that it often makes sense to try to fit the system into this form as a starting point.

In an ODE class, these equations are solved for two solutions:

Homogeneous	Particular
• input = 0	• depends on input
• free response	• forced response
• natural response	

1 Time domain solutions

The homogeneous solution is straightforward:

Let $y = Ae^{st}$
Then $\dot{y} = Ase^{st}$
And $\ddot{y} = As^2 e^{st}$
$\vdots$
$\Rightarrow$ substitute this assumed solution into the ODE and then solve for A and s.

Mass-spring-damper example

$m\ddot{y} + b\dot{y} + ky = 0$

$mAs^2e^{st} + bAse^{st} + kAe^{st} = 0$

$\Rightarrow ms^2 + bs + k = 0$ (this is the characteristic equation)

$s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$

There are two solutions:

$s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2-4mk}}{2m}$

$s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2-4mk}}{2m}$

The complete solution can be written as: $y = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

How do we get values for A_1 and A_2? You need to define the initial conditions:

$y(0) = A_1 + A_2$
$\dot{y}(0) = s_1 A_1 + s_2 A_2$

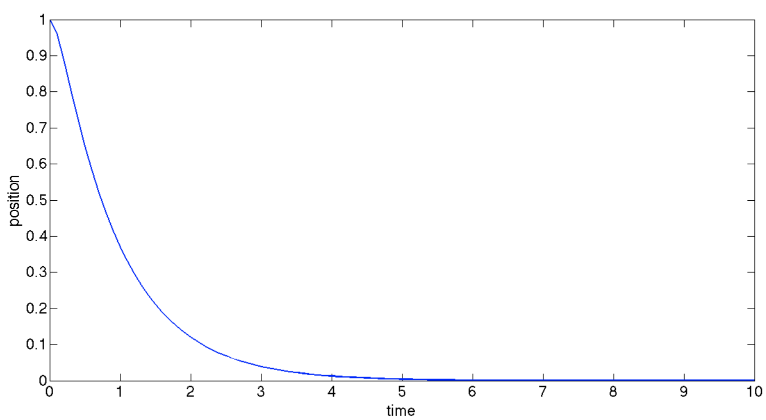
We have two equations and two unknowns, so you can solve for A_1 and A_2.

What do these solutions look like?

Here are two cases:

(a) If $b^2 > 4mk$, both s_1 and s_2 are negative and real

Here is a plot of $y = A_1 e^{s_1 t} + A_2 e^{s_2 t}$, where $s_1, s_2 < 0$ and real:

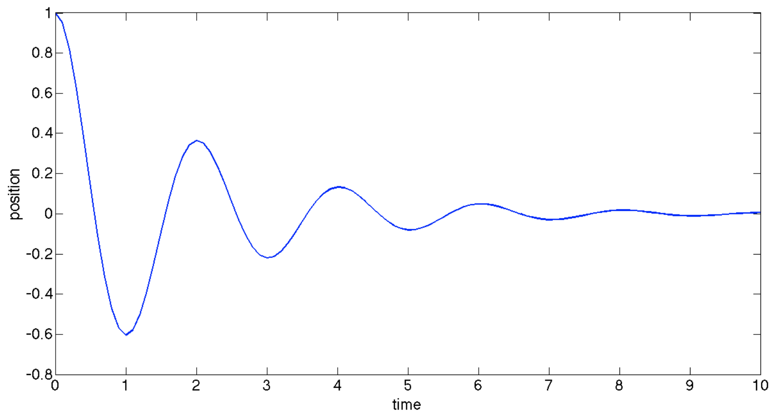



```
m = 0.1;
b = 1;
k = 1;
ydot0 = 0;
y0 = 1;
s1 = (-b + sqrt(b^2-4*m*k))/(2*m);
s2 = (-b - sqrt(b^2-4*m*k))/(2*m);
A1 = (ydot0-y0*s2)/(s1-s2);
A2 = (ydot0-y0*s1)/(s2-s1);
t = 0:0.1:10;
y = A1*exp(s1*t) + A2*exp(s2*t);
figure
plot(t,y)
xlabel('time')
ylabel('position')
```

(b) If $b^2 < 4mk$, s_1 and s_2 are <u>complex</u> (in fact, complex conjugates)

These will oscillate, since Euler's formula says that: $e^{j\theta} = \cos\theta + j\sin\theta$

Here is a plot of $y = A_1 e^{s_1 t} + A_2 e^{s_2 t}$, where s_1, s_2 are complex with negative real parts:

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```

- Is there another possible case?

- Solutions to equations of this form are always real roots or complex conjugate pairs. (Why?)

Now imagine there is a force applied to the mass, so the equation of motion becomes $m\ddot{y} + b\dot{y} + ky = f$. There are several ways to find the particular response to such a system using time-domain analysis, and these depend on the form of f. We will not review these techniques in this class because there is another solution technique that handles the homogeneous and particular solutions at once. It also provides greater insight about system structure. This is the Laplace transform.

2 Laplace transforms

A Laplace transform transforms:

1. a function of a real variable (like time) to a function of a complex variable (like s)

2. problems with differential equations to algebra

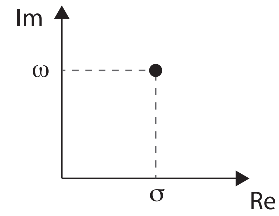
3. convolution to multiplication

The solution process for ODEs looks like:

$$\text{ODE} \quad \underset{\mathcal{L}}{\longrightarrow} \quad \begin{array}{c}\text{algebra problem}\\ \text{(sometimes stop here)}\end{array} \quad \underset{\mathcal{L}^{-1}}{\longrightarrow} \quad \text{solution of ODE}$$

Definition of the Laplace transform

Given a function $f(t)$,



$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(\tau)e^{-s\tau}d\tau$

s is a complex number: $s = \sigma + j\omega$

Superposition

It naturally follows that:

$\mathcal{L}[f_1(t) + af_2(t)] = F_1(s) + aF_2(s)$

Exponential

Since the exponential is such an important building block in our ODEs, it makes sense to look in more detail at the Laplace transform of the exponential:

$f(t) = Ae^{-\alpha t} \qquad t \geq 0$

$\mathcal{L}[f(t)] = \int_0^{\infty} Ae^{-\alpha\tau}e^{-s\tau}d\tau$

$= A\int_0^{\infty} e^{-(\alpha+s)\tau}d\tau$

$= \frac{-A}{\alpha+s}e^{-(\alpha+s)\tau}|_0^{\infty}$

$= \frac{-A}{\alpha+s}\left(e^{-(\alpha+s)\cdot\infty} - e^{-(\alpha+s)\cdot 0}\right)$

$= \frac{-A}{\alpha+s}(0-1) = \frac{A}{\alpha+s}$

Step function

$f(t) = A$ or $f(t) = A \cdot 1(t)$, where $1(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{otherwise} \end{cases}$

$\mathcal{L}[f(t)] = \int_0^{\infty} Ae^{-s\tau}d\tau$

$= \frac{A}{-s}e^{-s\tau}|_0^{\infty}$

$= \frac{A}{s}$

Notice that the exponential converges to a step as $\alpha \to 0$.

Differentiation

Here is the result:

$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$

This is a great example of integration by parts:

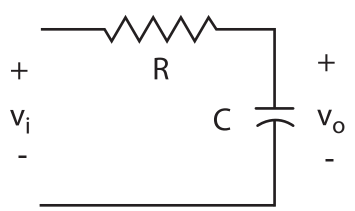
$F(s) = \int_0^\infty f(\tau)e^{-s\tau}d\tau = f(\tau)\frac{e^{-s\tau}}{-s}|_0^\infty - \int_0^\infty \left[\frac{df(\tau)}{d\tau}\right]\frac{e^{-s\tau}}{-s}d\tau$

(note that $\int u\,dv = u\,v - \int v\,du$)

$= -\frac{1}{s}\left[f(\infty)\cdot 0 - f(0) - \mathcal{L}\left\{\frac{df(t)}{dt}\right\}\right]$

$\Rightarrow sF(s) = f(0) + \mathcal{L}\left[\frac{df(t)}{dt}\right]$

RC circuit example



$\dot{v}_o = \frac{1}{RC}(v_i - v_o)$

$sV_o(s) - v_o(0) = \frac{1}{RC}(V_i(s) - V_o(s))$

$V_o(s)(RCs + 1) = V_i(s) + v_o(0)RC$

$V_o(s) = \frac{RC\,v_o(0)}{RCs+1} + \frac{V_i(s)}{RCs+1}$
(free/homogeneous and forced/particular)

We have just developed the *transfer function* for the RC circuit. Now we will solve for $v_o(t)$.

$v_o(t) = \mathcal{L}^{-1}[V_o(s)] = \mathcal{L}^{-1}\left[\frac{RC\,v_o(0)}{RCs+1}\right] + \mathcal{L}^{-1}\left[\frac{V_i(s)}{RCs+1}\right]$

$\uparrow v_o(0)e^{-\frac{t}{RC}}$

Particular solution for a step input $V_i(s) = \frac{A}{s}$:

$v_o(t) = v_o(0)e^{-\frac{t}{RC}} + \mathcal{L}^{-1}\left[\frac{\frac{A}{RC}}{s(s+\frac{1}{RC})}\right]$

$\uparrow \mathcal{L}^{-1}\left[\frac{A}{s} - \frac{A}{s+\frac{1}{RC}}\right] = A - Ae^{\frac{-t}{RC}}$

$\Rightarrow v_o(t) = v_o(0)e^{\frac{-t}{RC}} + A(1 - e^{\frac{-t}{RC}})$