ENGR 105: Feedback Control Design
Winter 2013

Lecture 5 - System Response

Wendesday, January 16, 2013

Today's Objectives

1. use partial fraction expansion to solve for the impulse response
2. solve for the response to general inputs
3. derive the Final Value Theorem

Reading: FPE Section 3.1

1 Solving for the Impulse Response

Writing a transfer function as a partial fraction expansion results in an equation of the following form, assuming that the input $U(s)$ is an impulse:

$$Y(s) = H(s) = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \cdots + \frac{C_n}{s-p_n}$$

The impulse response is given by:

$$y(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t} + \cdots + C_n e^{p_n t}$$

The C_i are known as residues and can be solved for using:

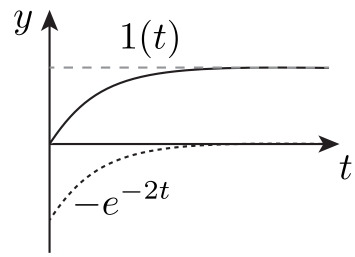
$$C_i = Y(s)(s-p_i)|_{s=p_i}$$

To see this, consider that

$$Y(s)(s-p_1)|_{s=p_i} = C_1 + \frac{C_2\overset{0}{\overbrace{(s-p_1)}}}{s-p_2} + \cdots + \frac{C_n\overset{0}{\overbrace{(s-p_1)}}}{s-p_n}$$

... if there are no repeated roots. (More on that shortly.)

Example



$Y(s) = H(s) = \frac{2}{s(s+2)} = \frac{C_1}{s} + \frac{C_2}{s+2}$

$C_1 = Y(s)s|_{s=0} = \frac{2}{s+2}|_{s=0} = 1$

$C_2 = Y(s)(s+2)|_{s=-2} = \frac{2}{s}|_{s=-2} = -1$

$\Rightarrow Y(s) = \frac{1}{s} - \frac{1}{s+2}$

$y(t) = 1(t) - e^{-2t}$

This is the impulse response of the system $H(s) = \frac{2}{s(s+2)}$.

Repeated poles

What if poles are repeated?

$Y(s) = \frac{C_1}{s-p_1} + \frac{C_2}{(s-p_1)^2} + \cdots + \frac{C_n}{s-p_n}$

$C_2 = Y(s)(s-p_1)^2|_{s=p_1}$

$C_1 = \frac{d}{ds}\left[Y(s)(s-p_1)^2\right]|_{s=p_1}$

$y(t) = C_1 e^{p_1 t} + C_2 t e^{p_1 t} + \cdots + C_n e^{p_n t}$
$\nearrow$ Is this okay?

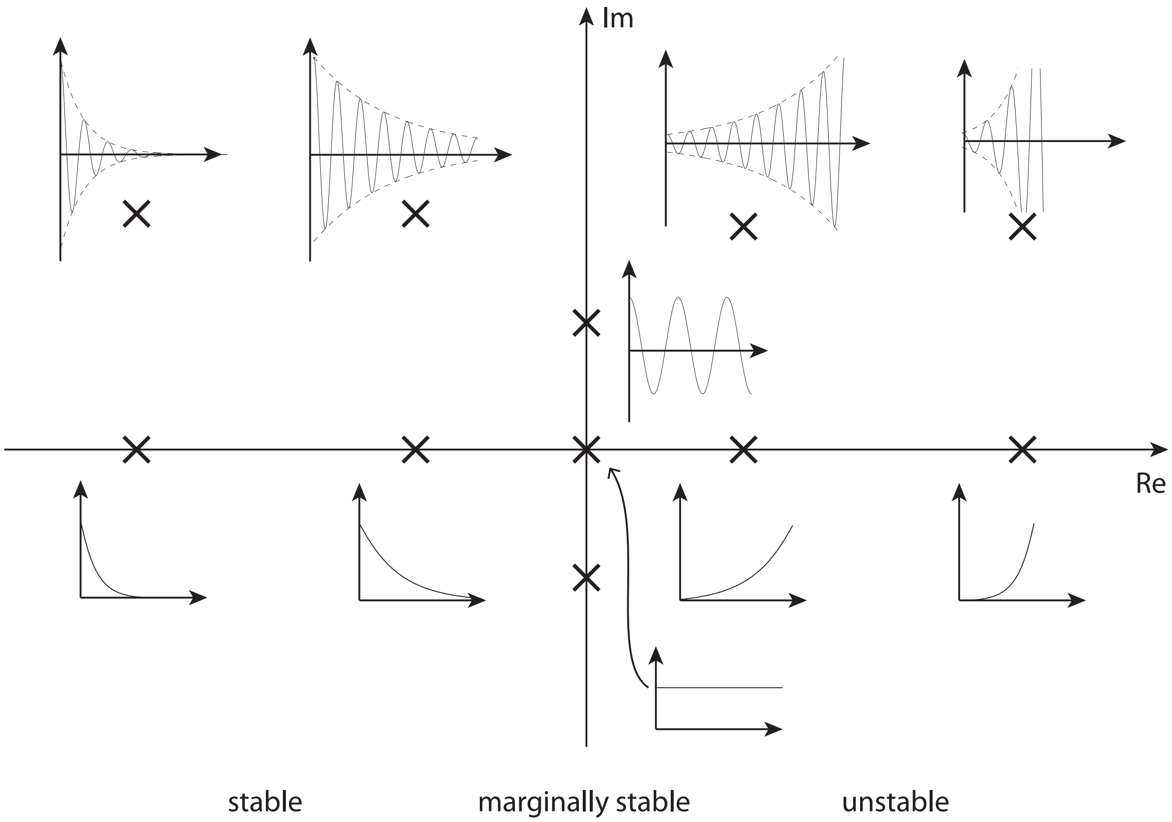
In general, for a repeated pole of multiplicity m,

$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^m}\right] = \frac{1}{(m-1)!} t^{m-1} e^{-at}$

Since $\lim\limits_{t\to\infty} \dfrac{t^{m-1}}{e^{at}} = 0 \quad \forall m$, the response dies out.

Thus, repeated *stable* poles are okay. It may grow quite large before it decays, however, suggesting that stability is not our only system requirement.

So if we have the transfer function, we can do a partial fraction expansion, look at the poles, and tell both stability and the building blocks of the response:



The response gets *faster* the farther you move from the origin on the real axis. The response gets *more oscillatory* the farther you move from the origin on the imaginary axis.

2 Response to general inputs

So far we looked at the impulse response, such that $Y(s) = H(s)$. The same techniques can be used to solve for the system response to any input $U(s)$.

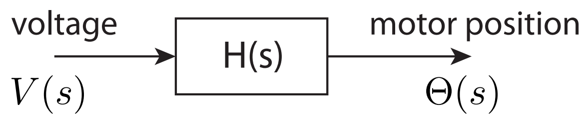
$$Y(s) = H(s)U(s) = \frac{K\prod^{m}(s-z_i)}{\prod^{n}(s-p_i)}\frac{K_u\prod^{m_u}(s-z_i)}{\prod^{n_u}(s-p_i)}$$

$$= \underbrace{\frac{a_1}{s-p_1} + \frac{a_2}{s-p_2} + \cdots + \frac{a_n}{s-p_n}}_{\text{poles from system}} + \underbrace{\frac{a_{n+1}}{s-p_{n+1}} + \cdots + \frac{a_{n+m_u}}{s-p_{n+m_u}}}_{\text{poles from input}}$$

We can solve for the system response in the same manner as described in the previous lecture, by taking the inverse Laplace transform. Keep in mind that the residues for a given input will be different from those calculated for the impulse response, so we have to solve for them again.

Example

$H(s) = \frac{1}{s(s+2)}$



This might be the response of a motor position to voltage, if the inductance is small.

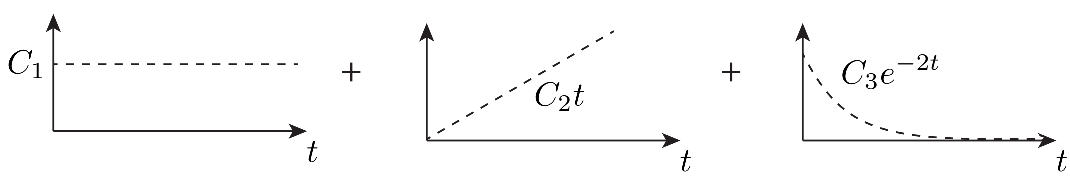
$\Theta(s) = H(s)V(s)$

What is the response to a step change in voltage? Let's use a step of 4 volts.

$V(s) = \frac{4}{s} \Rightarrow \Theta(s) = \frac{4}{s^2(s+2)}$

$\Theta(s) = \frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3}{s+2}$

We already have an idea of the qualitative response:



The response keeps growing in time (as we would expect). Mathematically, this happens because the input results in a double pole at $s = 0$. Multiple stable poles result in a stable system, but the same is not true for the marginally stable pole at the origin.

To get the response quantitatively, we need to solve for the residues:

$C_1 = \frac{d}{ds}\left[\Theta(s)s^2\right]|_{s=0} = \frac{d}{ds}\left(\frac{4}{s+2}\right)|_{s=0} = \frac{-4}{(s+2)^2}|_{s=0} = -1$

$C_2 = s^2\Theta(s)|_{s=0} = \frac{4}{s+2}|_{s=0} = 2$

$C_3 = (s+2)\Theta(s)|_{s=-2} = \frac{4}{s^2}|_{s=-2} = 1$

$\Rightarrow \theta(t) = -1 + 2t + e^{-2t}$

Poles at $s = 0$ act as integrators in the system. This should make sense: multiplication by s represents differentiation, and division by s represents integration.

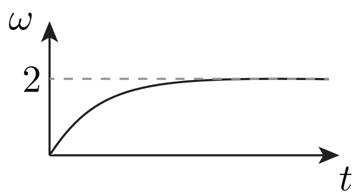
$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$

The integrator in this transfer function turns the step into a ramp, so the position of the motor keeps increasing. What about the motor velocity $\omega = \dot{\theta}$?

$\mathcal{L}[\omega(t)] = \Omega(s)$

$\Omega(s) = s\Theta(s) = \frac{4}{s(s+2)} = \frac{2}{s} + \frac{-2}{s+2}$

$\Rightarrow \omega(t) = 2(1 - e^{-2t})$



3 Final Value Theorem

If the system is stable, it is very easy to find the steady-state value that an input will produce.

From the derivative relationship in the Laplace transform:

$\mathcal{L}\left[\frac{dy}{dt}\right] = sY(s) - y(0) = \int_0^\infty e^{-st}\frac{dy}{dt}dt$

Taking the limit as $s \to 0$

$$\lim_{s\to 0}[sY(s) - y(0)] = \lim_{s\to 0}\int_0^\infty e^{-st}\frac{dy}{dt}dt$$
$$\lim_{s\to 0}[sY(s)] - y(0) = \int_0^\infty \frac{dy}{dt}dt = y(\infty) - y(0)$$
$$\Rightarrow \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

This is the *Final Value Theorem.* (Which only works if the system is stable!)

In the example above:

$$\Rightarrow \lim_{t\to\infty}\omega(t) = \lim_{s\to 0} s\Omega(s) = \lim_{s\to 0} s\frac{4}{s(s+2)} = 2$$

Thus, if we know the transfer function and the input, finding the steady-state value (if it exists) is extremely simple.